Competitive Equilibria in School Assignment

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Abstract

Top Trading Cycles was originally developed as an elegant method for finding a competitive equilibrium of Shapley and Scarf's housing market. We extend the definition of a competitive equilibrium to the school assignment problem and show that there remains a profound relationship between Top Trading Cycles and a competitive equilibrium. Specifically, in every competitive equilibrium with weakly decreasing prices, the equilibrium assignment is unique and exactly corresponds to the Top Trading Cycles assignment. This provides a new way of interpreting the worth of a student's priority at a given school. It also provides a new way of explaining Top Trading Cycles to students and a school board.

Key Words: Top Trading Cycles, School Choice, Competitive Equilibria.

JEL Classification: C78, D61, D78, I20

A hallmark of Lloyd Shapley's work is to introduce a seemingly simple model and to provide an elegant solution to the problem. A great part of his lasting legacy is that many years later these problems and solutions have been the basis for modern market design. A perfect example of this is school assignment. One of the seminal papers in market design is Abdulkadiroglu and Sönmez (2003) which introduced school assignment as a market design problem.[1] One can see the influence of Gale and Shapley (1962) and Shapley and Scarf (1974) throughout that paper. While Abdulkadiroglu and Sönmez's model is described as a variation of Gale and Shapley's College Admissions model, it can also be viewed as a generalization of the Shapley and Scarf (1974) house exchange model.[2] Moreover, the two solutions Abdulkadiroglu and Sönmez propose, the Deferred

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[1]Balinski and Sönmez (1999) is the first paper to consider centralized school assignment. Naturally, their paper is also heavily influenced by the work of Shapley.

[2]For example, Top Trading Cycles was first generalized for application to the object assignment problem by Papai (2000) and Abdulkadiroglu and Sönmez (1999).

Acceptance algorithm and the Top Trading Cycles algorithm (hereafter TTC), are both natural modifications of the algorithms introduced in Gale and Shapley (1962) and Shapley and Scarf (1974), respectively.

While Shapley and Scarf (1974) is primarily remembered for the introduction of TTC, this was not the paper's original purpose. Debreu and Scarf (1963) established the connection between the core and a competitive equilibrium under standard regularity conditions such as convexity of preferences, transferable utility, and perfect divisibility of the objects. Shapley and Scarf's stated purpose is to demonstrate that the same relationship between the core and a system of competitive prices can exist in a model where none of these assumptions hold. In their housing exchange model, there are n people and n houses. Each person owns one house and has only ordinal preferences over the other houses. Shapley and Scarf demonstrate that in this model, where none of the standard regularity conditions hold, a core assignment always exists and corresponds in a natural way to the outcome from a system of competitive equilibrium prices.

Many papers have studied the properties of TTC when applied to the school assignment problem. However, to the best of our knowledge, no paper has considered whether or not a competitive equilibrium exists in this more general model and if so, what its properties are. Stated differently, in the housing model it is not clear *a priori* the relative value of each house. However, this is revealed by solving for the competitive equilibrium. In the school assignment problem, no student owns a seat at a school. Instead, a student has a priority at each school. These priorities represent a claim on the school, and therefore result in some degree of "ownership" over the school. However, it is not clear *a priori* the value of a priority at a particular school.

In this paper we return to the classic concept of a competitive equilibrium in order to better understand the value of a particular priority at a particular school in the school assignment problem. Our approach is to assign a value to each priority. We allow each student to sell her most valuable priority and buy the best priority that she is able to afford (where the "best" priority is the priority that gains her admittance to the best possible school). We define an equilibrium to occur when the prices clear the market. Specifically, for each school a with capacity for q students, q priorities are sold, q priorities are purchased, and these priorities exactly coincide.

We demonstrate that a competitive equilibrium always exists for the school assignment problem (Theorem 1).[3] This result will not be surprising to readers familiar with TTC. Just as in the housing

[3] Miralles and Pycia (2014) consider the many-to-one assignment problem without transfers under the model of Hylland and Zeckhauser (1979). They establish the Second Welfare Theorem by showing that any Pareto efficient

market, we show that TTC can be used to find a competitive equilibrium of the school assignment problem. However, our main result is surprising. We show that all competitive equilibria where prices are weakly decreasing induce the same assignment: the assignment made by TTC (Theorem 2).

Interestingly, a similar result is true for the original Shapley and Scarf housing model. Roth and Postlewaite (1977) demonstrate that the unique assignment in the core is the assignment made by TTC. As any competitive assignment is in the core, this establishes that a student purchases the same house in any competitive equilibrium. However, the school assignment problem is significantly more complicated than the housing market problem. While there exists only one "reasonable" assignment for a housing problem, there exist several for the school assignment problem.[4] Therefore, it is surprising that monotonic equilibria have such a consistent structure. We demonstrate that alternative assignments can result from a competitive equilibrium when the worth of priorities are not monotonic (Example 1).

1 Model

We consider a finite set of students $I = \{i, j, k, \ldots\}$ and a finite set of schools $S = \{a, b, c, \ldots\}$. Each student $i \in I$ has a complete, irreflexive, and transitive preference relation P_i over $S \cup \{\emptyset\}$ where $\emptyset$ denotes the option of being unassigned. Here, $a \; P_i \; b$ indicates that student i strictly prefers school a to school b. Given P_i, we define the symmetric extension R_i by $a \; R_i \; b$ if and only if $a \; P_i \; b$ or $a = b$. A school a is **acceptable** for student i if $a \; P_i \; \emptyset$.

The capacity of each school $a \in S$ is given by q_a. Let $q_\emptyset = |I|$. Each school $a \in S$ has a complete, irreflexive, and transitive priority order $\succ_a$ over I. In particular, $i \succ_a j$ is interpreted as student i has a higher priority for school a than student j. We define $\succeq$ analogously to our definition of R. A school choice problem is defined as a list $(I, S, P, q, \succ)$ where $P = (P_i)_{i \in I}$, $q = (q_a)_{a \in S}$, and $\succ = (\succ_a)_{a \in S}$.

For each student i and school $a \in S$, we define the **rank** of i under $\succ_a$ to be the number of students with weakly higher priority than i:

$$rank(i, a) := |\, \{ j \in I \mid j \succeq_a i \} \,|.$$

assignment can be accomplished by a price mechanism.

[4] Consider, for example, the assignments made by Clinch and Trade or Prioritized Trading Cycles as described in Morrill (2015b) and (2013b).

An **assignment** is a function $\mu : I \to S \cup \{\emptyset\}$ such that for each $a \in S \cup \{\emptyset\}$, $|\{i \in I \mid \mu(i) = a\}| \leq q_a$. In a slight abuse of notation, for a set of students $I' \subseteq I$, we define $\mu(I') = \cup_{i \in I'} \mu(i)$ and $\mu(a) = \{i \in I \mid \mu(i) = a\}$ for each $a \in S$. An assignment μ is **nonwasteful** if there does not exist a school-student pair $(a, i) \in S \times I$ such that $a \ P_i \ \mu(i)$ and $|\mu(a)| < q_a$.

Under school choice problem $(I, S, P, q, \succ)$, the TTC mechanism selects its outcome through the following algorithm:

Round 1: Assign a counter to each school, and set it equal to its quota. Each student points to her most preferred, acceptable school. Each school with available seats points to the top-ranked student in its priority order. Since there are a finite number of students and schools, there must be at least one cycle. Assign each student in a cycle to the school she points to and remove her. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is removed. If all of a student's acceptable schools have been removed, remove the student and assign her to $\emptyset$.

Round $k > 1$: Each remaining student points to her most preferred, acceptable, remaining school. Each remaining school points to the remaining student with the highest priority. Assign each student in a cycle to the school she points to and remove her. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. If all of a student's acceptable schools have been removed, remove the student and assign her to $\emptyset$.

The mechanism terminates when all students have been assigned.

For convenience, we assume the total number of seats at schools is equal to the number of students (i.e., $\Sigma_{a \in S} q_a = |I|$) and that each student finds all schools acceptable. This assumption is made purely for expositional convenience, and all results continue to hold when it is relaxed. Under our assumption, TTC assigns each student to a school in S and each school in S fills its quota.

2 Competitive Equilibria

In a classic exchange market, a competitive equilibrium consists of a price for each object such that supply equals to demand. The school assignment problem is different in that no student owns a school; however, each school has priorities over the students. A typical design objective is to have students with higher priority at a school be more likely to gain admittance to that school than students with lower priority, i.e., a student with higher priority has higher claim for that school.

A key objective of this paper is to determine a systematic way of assigning value (price) to the priorities at each school. Typically, we expect the value of a priority to be determined both by the demand for the seats at a school and the supply of seats at the school. Both a low priority at a highly demanded school and a high priority at an underdemanded school should be worth very little. Therefore, it is natural to let the market forces of supply and demand determine the worth of each priority.

We denote the set of priorities by $Z \equiv \{(m, a) \mid m \in \{1, 2..., |I|\} \text{ } and \text{ } a \in S\}$ where (m, a) represents the m^{th} highest priority at school $a \in S$. We define a **priority value function** $\nu : Z \to \mathbb{R}_+$ where $\nu(m, a)$ represents the value of the m^{th} highest priority at school $a \in S$. The value function is our analog to prices. We define a priority value function to be **monotonic** if for each school $a \in S$, $\nu(x, a) \geq \nu(y, a)$ whenever $x < y$ (a priority is worth weakly more than any "lower" priority). Given priority value function ν, a student i's **wealth**, denoted by $\omega_i(\nu)$, is defined to be her most valuable priority:

$$\omega_i(\nu) := max_{a \in S} \text{ } \nu(rank(i, a), a).$$

Each student sells one of her priorities and buys one priority. Therefore, we define an **allocation** to be a function $\alpha : I \to Z \times Z$ where for each $i \in I$, $\alpha(i) = (\alpha^s(i), \alpha^p(i))$. Here, $\alpha^s(i) \in \{(rank(i, a), a) \mid a \in S\} \subseteq Z$ denotes the priority that i has sold. Similarly, $\alpha^p(i) \in Z$ denotes the priority that student i has purchased.[5] An allocation α is **feasible** if $\cup_{i \in I} \alpha^s(i) = \cup_{i \in I} \alpha^p(i)$ (every priority that is bought is also sold).

Given an allocation α, we induce an assignment in a natural way. If a student i purchases priority $(x, a) \in Z$, i.e., $\alpha^p(i) = (x, a)$, she is assigned to school a.[6] To avoid a triviality, we further impose that she only is assigned to school a if $\nu(x, a) > 0$.[7] The market clearing conditions are that each student is assigned to a school and each school a is assigned q_a students.[8]

As with a classical competitive equilibrium, we define a students budget set and demand. For a given value function ν, a student i's **budget set** is defined as: $B_i(\nu) = \{(x, a) \in Z \mid \nu(m, a) \leq \omega_i(\nu)\}$. Given value function ν, we define the set of **affordable schools** as $A_i(\nu) = \{a \in S \mid \exists(x, a) \in$

[5]The definition of allocation can be easily extended to the general case. In particular, we can define allocation as $\alpha : I \to Z \cup \{\emptyset\} \times Z \cup \{\emptyset\}$ where $\alpha^s(i) = \emptyset$ means i does not sell a priority and $\alpha^p(i) = \emptyset$ means i does not purchase a priority.

[6]For the general case, if $\alpha^p(i) = \emptyset$, then she is unassigned.

[7]Otherwise, for any school a, i could always purchase $(rank(i, a), a)$ and be assigned to a.

[8]In the general case, the two conditions would be that the induced assignment is a proper assignment and that it is nonwasteful.

$B_i(\nu)$ such that $0 < \nu(x, a)\}$. As a reminder, a student is admitted to a school if she buys a priority with non-zero value. In words, the affordable schools for student i are the schools she can buy admittance to. We call i's most preferred school in $A_i(\nu)$ (possibly $\emptyset$) her **favorite affordable school**. If i's favorite affordable school is $a^* \neq \emptyset$, then we define i's **demand** to be $\{(x, a^*) \in Z | 0 < \nu(x, a^*) \leq \omega_i(\nu)\}$. In the degenerate case where i cannot afford any school (i's favorite affordable school is $\emptyset$), then we define i's demand to be $\{(x, a) \in Z | \nu(x, a) = 0\}$.[9]

Definition 1. A **competitive equilibrium** is a feasible allocation, α, and priority value function, ν, such that

1. Each student sells one of her most valuable priority, $\nu(\alpha^s(i)) = \omega_i(\nu)$, and purchases a priority in her demand.

2. In the induced assignment, each student is assigned to a school and no school a is assigned to more than q_a students.[10]

A competitive equilibrium (α, ν) is **monotonic** if the priority value function ν is monotonic. That is to say, if the worth of a priority is weakly decreasing in the rank of a student ($\nu(x, a) \geq \nu(y, a)$ for any $x < y$).

Note that in any competitive equilibrium, assigning a student to the school where she bought a priority is a valid assignment. We refer to this as the **competitive assignment**. Also, if j purchases i's priority at s, then we will say i sold her priority to j. Our competitive equilibrium is purely a system of exchanges, but we are now able to interpret what assignment of values to priorities makes this system of exchanges possible.

Now, we are ready to present our existence result.

Theorem 1. *In any school choice problem $(I, S, P, q, \succ)$, there always exists a monotonic competitive equilibrium.*

Proof. Just as in Shapley and Scarf (1974), TTC can be utilized in a natural way to find a monotonic competitive equilibrium. We consider an implementation of TTC under problem $(I, S, P, q, \succ)$ where in each round all cycles are processed simultaneously.[11] Let S^k denote the cycles processed

[9] Note that this set is nonempty as all of i's priorities have value zero or else she would have a nonempty set of affordable schools. By assumption, each student considers all schools acceptable.

[10] In the general case, this condition is that the induced assignment is a proper assignment and that it is nonwasteful.

[11] The TTC mechanism is defined in Section 1.

in the k^{th} round of TTC and K be the last round of TTC under problem $(I, S, P, q, \succ)$. As a reminder, we have assumed for expositional convenience that $\Sigma_{a \in S} q_a = |I|$ and that all students find all school acceptable.

Fix a positive real number $\pi^1 \in \mathbb{R}_{++}$. Let $(i_1, a_1, \ldots, i_n, a_n)$ be a cycle in the first round of TTC. For each $1 \leq k \leq n$, set $\nu(rank(i_k, a_{k-1}), a_{k-1}) = \pi^1$ and $\alpha(i_k) = ((1, a_{k-1}), (1, a_k))$.[12] In words, i_k sells her priority at a_{k-1} (the school pointing at her) and buys the highest priority at a_k (the school she is pointing to). Note that by the definition of TTC, a_k is her favorite school and therefore in her demand. For each school $a \in S$, if the number of priorities allocated reaches to q_a, then we set the values of the unused priorities at school a to 0.

Similarly, fix a positive real number $\pi^2 \in \mathbb{R}_{++}$ such that $\pi^2 < \pi^1$. Let $(i_1, a_1, \ldots, i_n, a_n)$ be a cycle in the second round of TTC. For each $1 \leq k \leq n$, set $\nu(rank(i_k, a_{k-1}), a_{k-1}) = \pi^2$ and $\alpha(i_k) = ((rank(i_k, a_{k-1}), a_{k-1}), (rank(i_{k+1}, a_k), a_k))$. In words, i_k sells her priority at a_{k-1} and purchases i_{k+1}'s priority at a_k. If there is a school a_k and a priority (x, a_k), where $x < rank(i_{k+1}, a_k)$, but (x, a_k) has not yet been assigned a value, then set $\nu(x, a_k) = \pi^1$. Note that by the definition of TTC, since a_k points at i_{k+1}, the x^{th} ranked student at a_k was already assigned in the first round of TTC. By construction, for any student i_k in the cycle, i_k's wealth is π^2 and i_k cannot afford any school she prefers to a_k. For each school $a \in S$, if the number of priorities allocated reaches to q_a, then we set the values of the unused priorities at school a to 0.

Proceed in this manner letting $\pi^k \in \mathbb{R}_{++}$ denote the value assigned to each priority in S^k where $\pi^1 > \pi^2 > \ldots > \pi^k > \ldots > \pi^K > 0$. Then, set the values of the priorities unused when the mechanism terminates to 0.

Note that the values of priorities of schools in S are (weakly) decreasing. Further, each student's most valuable priority is the one used by TTC. Therefore, if there is a school a student prefers to her TTC assignment, then that school was assigned to capacity in a previous round and as a result, she cannot afford any of that school's non-zero priced priorities. Therefore, her TTC assignment is her favorite choice that she can afford. Indeed, each student who is assigned to a school in S spends all of her wealth when she purchases a priority from that school. Therefore, these values constitute a monotonic competitive equilibrium. □

Of course, no problem has a unique monotonic competitive equilibrium as any monotonic transfor-

[12] Throughout this proof, it is understood that for $k = 1$, $a_{k-1} = a_n$. Also, notice that $\alpha(i_k) = ((rank(i_k, a_{k-1}), a_{k-1}), (rank(i_{k+1}, a_k), a_k))$.

mation of any monotonic equilibrium values (prices) is itself a monotonic competitive equilibrium. However, we demonstrate that the assignment in a monotonic competitive equilibrium is always unique and exactly corresponds to the TTC assignment.

Theorem 2. *When values (prices) are monotonic, there exists a unique competitive equilibrium assignment. In particular, in any monotonic competitive equilibrium, each student purchases a priority at the school she is assigned to by TTC.*

In order to prove Theorem 2, we benefit from the following intermediate lemma.

Lemma 1. *In a competitive equilibrium every student spends her entire wealth.*

Proof. Let values for priorities ν and feasible allocation $\alpha = (\alpha^s, \alpha^p)$ constitute a competitive equilibrium.

Each purchased priority must be affordable. Therefore,

$$\sum_{i \in I} \nu(\alpha^p(i)) \leq \sum_{i \in I} \omega_i(\nu).$$

If some student did not spend her entire wealth, then

$$\sum_{i \in I} \nu(\alpha^p(i)) < \sum_{i \in I} \omega_i(\nu).$$

However, each student's wealth equals the amount she sold her priority for. Therefore,

$$\sum_{i \in I} \omega_i(\nu) = \sum_{i \in I} \nu(\alpha^s(i)).$$

Finally, there is only one price for any priority. Therefore, the sum of the purchase prices must equal the sum of the sale prices. Moreover, feasibility of α, i.e. $\alpha^p(I) = \alpha^s(I)$, implies that

$$\sum_{i \in I} \nu(\alpha^p(i)) = \sum_{i \in I} \nu(\alpha^s(i)).$$

Hence,

$$\sum_{i \in I} \nu(\alpha^p(i)) = \sum_{i \in I} \omega_i(\nu).$$

Therefore, it would be a contradiction if some student did not spend her entire wealth. □

Now we are ready to prove Theorem 2.

Proof. Consider a competitive equilibrium (α, ν) where the values for priorities ν are monotone and $\alpha = (\alpha^s, \alpha^p)$ is a feasible allocation. Recall that under our assumption in any competitive equilibrium each student purchases a priority from a school in S. Let π^1 be the value of the most expensive priority, let $S_1 = \{(rank(i,a),a)|\nu(rank(i,a),a) = \pi^1 \; for \; all \; a \in S \; and \; i \in I\}$ be the set of most expensive priorities, and let I_1 be the set of students that purchase a priority for price π^1. By the definition of the competitive equilibrium and Lemma 1, $\pi^1 > 0$ and $I_1 \neq \emptyset$. We show by induction that I_1 can be partitioned into top trading cycles.[13] For the base step, consider any $j \in I_1$ and have j point to her favorite school $a \in S$. Since j purchases the most expensive priority, j can afford any priority, and therefore j must purchase a priority at a. Some student, possibly j herself, sells a priority to j for the price π^1. Label the student with the highest priority at a with k. Since values are monotonic, k must have wealth at least π^1. Since each student spends her entire wealth (Lemma 1) and π^1 is the price of the most expensive priority, k must have wealth exactly equal to π^1 and must purchase a priority for price π^1, and therefore $k \in I_1$. We continue the process of taking a student in I_1, pointing to her favorite school (she must purchase a priority at that school), and then pointing to the student with the highest priority at that school, and so on. Due to finite number of students and schools, eventually we must repeat a student. This is exactly a top trading cycle, and each student in the cycle is a member of I_1.

Now suppose that we have removed some subset of students $I_1' \subseteq I_1$ who have formed trading cycles and consider a student $\hat{j} \in I_1 \setminus I_1'$, if there exists any. As before, have $\hat{j}$ point to her favorite school with remaining capacity, $\hat{a}$. Note that $\hat{a}$ is $\hat{j}$'s favorite school as $\hat{j}$ can afford any priority. Some student, possibly $\hat{j}$ herself, sells a priority to $\hat{j}$ for the price π^1. Label the student in $I_1 \setminus I_1'$ with the highest priority at $\hat{a}$ with $\hat{k}$. By Lemma 1 and the arguments explained for student k, $\nu(rank(\hat{k},\hat{a}),\hat{a}) = \pi^1$ and $\hat{k} \in I_1$. By following the same argument we can represent all students in I_1 via top trading cycles and they are assigned to their most preferred school. Therefore, each student who purchases a priority that costs π^1 receives the same assignment as in TTC.

Similarly, let π^2 be the value of the most expensive priority of students in $I \setminus I_1$, let $S_2 = \{(rank(i,a),a)|\nu(rank(i,a),a) = \pi^2 \; for \; a \in S \; and \; i \in I \setminus I_1\}$. Note that, if $\pi^1 \geq \nu(rank(i',a),a) > \pi^2$, then $i' \in I_1$ and by the definition of the competitive equilibrium the priorities with price between π^1 and π^2 are not sold or bought. Let I_2 be the set of students that purchase a priority for price π^2. By Lemma 1, $I_2 \neq \emptyset$. Consider any $j \in I_2$ and have j point to her favorite school

[13] We say a cycle is a top trading cycle if it occurs under TTC.

among the ones with remaining seats after the students in I_1 and their assigned seats are removed. Denote that school with a. Since j purchases the most expensive priority among the remaining ones, and a is her favorite remaining school, j must purchase a priority at a. Some student in $I \setminus I_1$, possibly j herself, sells a priority to j for the price π^2. Label the student with the highest priority at a among the ones in $I \setminus I_1$ with k. Since values are monotonic, k must have wealth at least π^2. Since each student spends her entire wealth (Lemma 1) and π^2 is the price of the most expensive remaining priority, k must have wealth exactly equal to π^2 and must purchase a priority for price π^2, and therefore $k \in I_2$. We continue the process of taking another student in I_2, pointing to her favorite school (she must purchase a priority at that school), and then pointing to the student with the highest priority at that school, and so on. Due to finite number of students and schools, eventually we must repeat a student. This is exactly a top trading cycle, and each student in the cycle is a member of I_2. By following the same argument we can represent all students in I_2 via top trading cycles and they are assigned to their most preferred school among the remaining ones once the students in I_1 together with their assignments are removed. Therefore, each student who purchases a priority that costs π^2 receives the same assignment as in TTC.

By following the same way for the remaining students, we have the desired result. □

We show by example that when the prices of priorities are not weakly decreasing, then there is no longer a unique competitive equilibrium assignment. This example is taken from Morrill (2015b).

Example 1. Suppose there are three students $I = \{i, j, k\}$ and two schools $S = \{a, b\}$. School a has a capacity of two while b has a capacity of one. Define P and $\succ$ according to the following rank-order lists:

P_i	P_j	P_k	$\succ_a$	$\succ_b$
b	a	b	i	j
a	b	a	j	k
$\emptyset$	$\emptyset$	$\emptyset$	k	i

Consider the following prices for priorities.

a	b
50	100
100	60
0	0

Consider the allocation where j purchases and sells the second priority at a; k purchases and sells the second priority at b; and i purchases and sells the first priority at a. Each student sells her most valuable priority, and each student purchases the best priority she can afford. Therefore, this constitutes a competitive equilibrium. However, this assignment differs from the TTC assignment which assigns i, j, and k to b, a, and a, respectively.

In Example 1 student j has one of the top q_a priority at school a. Hence, in order to be assigned to a j does not need to trade with i. Since a different set of trades occur, a different assignment is made.

3 Conclusion

The existence of a competitive equilibrium allows for a natural interpretation of the TTC assignment.[14] For example, it is well known that a student may have justified envy under the TTC assignment (Abdulkadiroglu and Sönmez, 2003).[15] This has widely been interpreted as the TTC assignment being unfair. However, the interpretation under a competitive equilibrium is quite different. Here, we consider what a priority is worth. A priority at a school is only valuable if it gains you admittance to that school *and* there is excess demand for the school. If Bowling Green High School is ranked first by all students but has capacity for only 200 freshman, then having the 201^{st} priority has the same worth as having the 800^{th} priority. More precisely, neither of these priorities have any worth at all as neither are high enough to gain admittance to the school. Similarly, if all students rank Perrysburg high school last, then having the highest priority at Perrysburg has no worth. Any student may attend Perrysburg if she chooses. Rather, the value of priorities are determined by the familiar forces of supply and demand.

TTC mechanism has been criticized since it does not totally eliminate priority violations. Consider the case where a student i is assigned to a by TTC, but there is a student j who desires a and has a higher priority at a. Is this fair? In a competitive equilibrium, we have a different interpretation.

[14]There exist several characterizations of TTC for the school assignment problem (see Abdulkadirgolu and Che, 2010; Morrill, 2015a; Morrill, 2013a; and Dur, 2014). However, several of these characterizations only hold when schools can be assigned to only one student (Abdulkadiroglu and Che, 2010, and Morrill 2013) and the general characterizations in Morrill (2015a) and Dur (2014) utilize at least one technical condition that may be difficult for a school board to interpret. In a recent and independent work, Leshno and Lo (2017) provide a characterization of TTC mechanism by using a continuum model. In particular, they show that TTC assignment can be described by n^2 admission thresholds. Similar to this current work, the threshold matrix and the priorities together can be interpreted as prices and endowments in a Walrasian economy.

[15]Student i is said to have *justified envy* if i is not matched to school s, i prefers s to her assignment, and i has higher priority at s than one of the students assigned to s.

Neither i nor j's priority at a is particularly valuable as neither has a high enough priority to gain them admittance to a. However, i is willing to give up her claim to a school that is just as sought after as a. In contrast, student j has no priority at any school that is valuable (or at least valuable enough). In terms of what the student gives up, the assignment of i instead of j to a is quite fair; i is willing to give up something of more value than j is able.

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