An Alternative Characterization of the Deferred Acceptance Algorithm

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Abstract

Kojima and Manea (2010) present two characterizations of when an allocation rule corresponds to the agent-proposing deferred acceptance algorithm for some substitutable priority rule of the objects being assigned. Building on their results we characterize when an allocation rule is outcome equivalent to the deferred acceptance algorithm for every substitutable priority rule. In particular, an assignment rule satisfies *mutual best* if an agent is always assigned her most preferred object whenever she has the highest priority for it. This mild requirement is a necessary but far from sufficient condition for an assignment rule to be stable. We demonstrate that any allocation mechanism that satisfies mutual best along with non-wastefulness, population monotonicity and either individually rational monotonicity or weak Maskin monotonicity not only is a stable assignment mechanism but is equivalent to the agent proposing deferred acceptance algorithm.

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1 Introduction

A recent paper by Kojima and Manea (2010), hereafter K&M, presents an axiomatic characterization of when an allocation rule corresponds to the agent-proposing deferred acceptance algorithm for some substitutable priority rule of the objects being allocated. Building on their results, we characterize when an allocation rule is outcome equivalent to the deferred acceptance algorithm for every substitutable priority rule.

K&M provide two characterizations of which their second is perhaps more intuitive. An allocation rule is non-wasteful if whenever an agent prefers an object to her assignment, the preferred object has been allocated its quota of agents. An allocation rule is population monotonic if whenever the set of agents being assigned is reduced, all remaining agents are made weakly better off. Finally, a set of preferences R' monotonically transforms preferences R at assignment μ if any object preferred to μ under R' is also preferred to μ under R. An allocation rule ϕ satisfies weak Maskin monotonicity if whenever R' is a monotonic transformation of R at $\phi(R)$, then every agent weakly prefers $\phi(R')$ to $\phi(R)$ under R'. Remarkably, K&M demonstrate that any allocation rule satisfying weak Maskin monotonicity, population monotonicity, and non-wastefulness is equivalent to the deferred acceptance rule for some substitutable priority rule.

Our paper provides a characterization of the deferred acceptance rule for every substitutable priority rule. An assignment mechanism satisfies *mutual best (MB)* if whenever an agent has highest priority at her top choice, then she is assigned her top choice. This weak assumption is clearly necessary but far from sufficient for an allocation to be stable. Surprisingly, MB is enough to strengthen either characterization introduced by K&M from being equivalent to the deferred acceptance algorithm for at least one substitutable priority rule to being equivalent to the deferred acceptance algorithm for every substitutable priority rule.

To emphasize the distinction between K&M's characterization and the current paper's contribution we compare the serial dictatorship[1] to the deferred acceptance algorithm. In a serial dictatorship, each agent is assigned a number and the agent with the highest number is allowed to choose her most preferred object. The agent with the second highest number chooses from the remaining objects, and so on. This mechanism satisfies all of K&M's axioms. It is both weakly Maskin and IR monotonic, it is non-wasteful, and it is population monotonic. Indeed, the serial dictatorship is equivalent to the deferred acceptance algorithm for one priority rule over the objects. Namely, if the agent with highest number has highest priority for every object, the agent with the second highest number has second highest priority for every object, and so on. However, except for this particular instance of priorities, the serial dictatorship bears little resemblance to the deferred acceptance algorithm. Since the serial dictatorship completely disregards the priorities of the objects, it is only a coincidence when the two mechanisms produce the same assignment.

This paper provides an alternative to characterizing the deferred acceptance algorithm in terms of stability. Gale and Shapley (1962) show that the deferred acceptance algorithm is the unique stable assignment that is weakly preferred by the students to all other stable assignments. Alcalde and Barbera (1994) characterize the deferred acceptance algorithm as the unique stable and strategy-proof mechanism. Balinski and Sonmez (1999) demonstrate the deferred acceptance algorithm is the unique allocation rule that is stable and respects improvements.[2] Our result is most similar to K&M's Lemma 2 which demonstrates the deferred acceptance algorithm is the unique mechanism that satisfies stability and weak Maskin monotonicity. The key difference is that our condition of MB is much weaker than stability. An-

[1]For a detailed discussion of serial dictatorships, see Abdulkadiroglu and Sonmez (1998).

[2]An allocation rule respects improvements if an agent is made weakly better off whenever her priority for every object weakly increases.

other paper that is closely related to K&M is Ehlers and Klaus (2009). They provide several alternative characterizations of the deferred acceptance algorithm when priorities are restricted to be responsive.

2 Framework

We use the framework from K&M with the notable exception that the mechanisms we consider are a function of both the preferences of the agents and the priority structure for each object being assigned. We fix a set of *agents* N along with a set of *objects* O. There exists one *null object*, denoted $\emptyset$, with the interpretation that being assigned the null object is equivalent to the agent being unassigned. Each object $a \in O$ has a quota q_a and $q_\emptyset = |N|$. An *allocation* $\mu = (\mu_i)$ is a vector such that for every $i \in N$, $\mu_i \in O \cup \{\emptyset\}$ and for each $a \in O \cup \{\emptyset\}$, $|\{i \in N | \mu_i = a\}| \leq q_a$. We let $\mu_a = \{i \in N | \mu_i = a\}$.

Each agent i has a *preference relation* R_i over all types $O \cup \{\emptyset\}$. R_i is strict, complete, transitive, and antisymmetric. P_i denotes the agents strict preferences over the objects, aP_ib if and only if aR_ib and $a \neq b$. An object is *acceptable* if it is preferred to the null object. $R = (R_i)_{i \in N}$ denotes the preference profile of all agents, and $R_{N'} = (R_i)_{i \in N'}$ denotes the preferences of any subset $N' \subset N$. We also adopt the notation $R_{-M} = R_{N \setminus M}$ and $R_{-i} = R_{N \setminus \{i\}}$. We write $\mu R \mu'$ if and only if $\mu_i R_i \mu'_i$ for all $i \in N$.

A *priority* for an object $a \in O$ is a correspondence $C_a : 2^N \to 2^N$, such that $C_a(N') \subseteq N'$ and $|C_a(N')| \leq q_a$ for all $N' \subseteq N$. Intuitively, $C_a(N')$ is the set of agents that a "chooses" when given the choice of any agent in N'. In the context of school choice, $C_a(N')$ is interpreted to be the students with highest priority at school a. The priority C_a is substitutable if for every $i \in N'' \subset N' \subset N$, if $i \in C_a(N')$ then $i \in C_a(N'')$. The priority C_a is *acceptant* if for all $N' \subseteq N$, $|C_a(N')| = \min\{|N'|, q_a\}$.

An assignment μ is individually rational if for every student i, $\mu_i R_i \emptyset$. An assignment is *blocked* if there exist a student i and a school a such that $aP_i\mu_i$ and $i \in C_a(\mu_a \cup i)$. We say i and a form a *blocking pair*. If there does not exist a blocking pair, then the allocation is *stable*.

We denote by $\mathcal{R}$, $\mathcal{C}$, and $\mathcal{A}$ the sets of possible preference relations, priority rules, and assignments, respectively. An *allocation rule* is a function $\phi : \mathcal{R} \times \mathcal{C} \to \mathcal{A}$. An allocation rule ϕ is *stable* if $\phi(R, C)$ is a stable allocation for all $R \in \mathcal{R}, C \in \mathcal{C}$. A particular important mechanism and the focus of this paper is Gale and Shapley's (1962) *deferred acceptance* algorithm. The student proposing version of the algorithm is as follows:
Step 1. Each student proposes to her most preferred, acceptable school. Each school tentatively accepts the group of students with highest priority (alternatively, the most preferred group) among those students that have proposed to it. The school rejects all other students.
Step i. Each student that was rejected in the previous round proposes to her most preferred school among those that are acceptable and have not yet rejected the student. Each school tentatively accepts the highest priority group of students among those that have applied and those that were tentatively accepted in the previous round. The school rejects the other applicants.

The deferred acceptance algorithm concludes when there are no new proposals from students. When school priorities are substitutable, the algorithm results in a stable assignment that is weakly preferred by every student to any other stable assignment.[3] The *deferred acceptance rule*, $DA(R, C)$ outputs the assignment that is generated when the deferred-acceptance algorithm is applied to (R, C).

The key difference between this framework and that of K&M is that except for one section of their paper the priorities of the objects are not primitive to the model. Part of what is remarkable about their results is that they

[3]See Roth and Sotomayor (1990).

are able to characterize when an assignment rule corresponds to the deferred acceptance algorithm using priority-free conditions.

R'_i is an *individually rational monotonic transformation* of R_i at $a \in O \cup \{\emptyset\}$ (R'_i i.r.m.t. R_i at a) if any object that is preferred to both a and $\emptyset$ under R'_i is preferred to both a and $\emptyset$ under R_i. R' is an IR monotonic transformation of R at an allocation μ (R' i.r.m.t. R at μ) if R'_i i.r.m.t. R_i at μ_i for all $i \in N$. An allocation rule ϕ satisfies *individualy rational monotonicity* (IR monotonicity) if R' i.r.m.t R at $\phi(R)$ implies $\phi(R')R'\phi(R)$. An allocation rule ϕ is *non-wasteful* if $aP_i\phi_i(R)$ implies $|\phi_a(R)| = q_a$. K&M demonstrate that that an allocation rule is the deferred acceptance rule for some acceptant substitutable priority rule if and only if it satisfies IR monotonicity and non-wastefulness.

For their second characterization, they define R'_i to be a *monotonic transformation* of R_i at $a \in O \cup \{\emptyset\}$ (r'_i m.t. R_i at a) if all objects ranked above a under R'_i are ranked above a under R_i. R' is a monotonic transformation of an allocation rule μ (R' m.t. R at μ) if R'_i m.t. R_i at μ_i for all $i \in N$. An allocation rule ϕ satisfies *weak Maskin monotonicity* if R' m.t. R at $\phi(R)$ implies $\phi(R')R'\phi(R)$. For any $N' \subseteq N$ and any preference profile R, let $(R_{N'}, R^{\emptyset}_{N \setminus N'})$ denote the preference profile that leaves R_i unchanged for each $i \in N'$ and ranks $\emptyset$ first for each $i \in N \setminus N'$. This is equivalent to considering the assignment problem restricted to the subset N'. An allocation rule ϕ is *population monotonic* if $\phi_i(R_{N'}, R^{\emptyset}_{N \setminus N'}, C)R_i\phi_i(R, C)$ for every $i \in N', N' \subseteq N, R \in \mathcal{R}$, and $C \in \mathcal{C}$. IR monotonicity implies both weak Maskin monotonicity and population monotonicity, but population monotonicity and weak Maskin monotonicity are possibly easier to interpret. For their second characterization, they demonstrate that an allocation rule ϕ is the deferred acceptance rule for some acceptant substitutable priority C if and only if ϕ satisfies non-wastefulness, weak Maskin monotonicity, and population monotonicity.

3 An extension of the deferred acceptance axioms

K&M's results are remarkable in that their conditions are priority-free and yet powerful enough to guarantee that the mechanism corresponds to the deferred acceptance rule for some priority rule. However, this comes at a cost. Their characterization only guarantees that the assignment corresponds to the deferred acceptance rule for *some* priority rule. It does not guarantee that the rule is equivalent to the deferred acceptance rule for the relevant priority rule or even if it is, whether or not it will continue to be equivalent to the deferred acceptance rule if the priority rule is changed.

In this section, we characterize the allocation rules that are equivalent to the deferred acceptance algorithm for *every* substitutable priority rule. This is critical in the context of a real-world assignment problem. Any assignment rule being implemented must be adaptable to changes in the priority rule for the objects. This is obvious if the priority rule represents the preferences of the objects and these preferences are at the discretion of the objects being matched, such as in the doctor-hospital resident match. However, this is equally true if the priority rule is fixed such as in the application to Boston public schools. A school district may change the priority rules from year to year. Moreover, the student body changes every year.

Surprisingly, we require only a mild assumption.

Definition 1. *Let N^* be the set of agents that find some object acceptable. An assignment mechanism $\phi(R, C)$ satisfies* ***mutual best (MB)*** *if aP_ib for every $b \in O \setminus \{a\}$ and $i \in C_a(N^*)$ imply $\phi_i(R, C) = a$.*

Any mechanism that violates MB violates stability and is therefore not equivalent to the deferred acceptance algorithm. However, MB is far from a sufficient condition for stability. For example, consider the Boston mechanism.[4] In the first round of the Boston mechanism, each school accepts up to its capacity the highest priority students among those that have ranked it first. Those schools that are at capacity are removed. In the second round, each school accepts up to its capacity the highest priority students among those students that have ranked it first among the remaining schools. Again, all schools that are at capacity are removed, and the process continues until all students are assigned.

The Boston mechanism satisfies MB as a student that has highest priority at her most preferred school will be accepted in the first round. However, the Boston mechanism need not be stable. For example, suppose there are three students i, j, and k, two schools A and B, and each school has a capacity of one student. Consider the following strict orders for schools:

$\succ_A$	$\succ_B$
i	j
j	i
k	k

and the following preferences of the students:

R_i	R_j	R_k
B	B	A
A	A	B

The Boston mechanism lasts only one round and assigns j to B, k to A, and leaves i unassigned. However, i and A block this assignment.

MB is a basic and clearly desirable property of an assignment rule; however, there are several interesting examples of mechanisms that violate it.

[4]See Abdulkadiroglu and Sonmez (2003). Other examples of non-stable algorithms that satisfy MB are the Newcastle, Birmingham, and Edinburgh algorithms described in Roth (1991).

Both the National Football League draft and the procedure to assign Naval Acadamy graduates to their initial military position violate MB.[5] Another interesting rule that violates MB is the linear programming matching procedure described in Roth (1991). This assignment rule was used to assign medical students to positions at the London Hospital and the University of Cambridge. Each student and hospital submits a rank ordered list of preferences, and using these lists, each pairing is assigned a numerical weight. For example, if a student and hospital rank each other first, a (1,1) pairing, then that match was assigned a weight of 40. Similarly, a (1,2) or (2,1) pairing was given a weight of 34 points, and so on. A linear programming procedure then determines an assignment that maximizes the sum of the weights of the matches. As Roth illustrates with an example, this procedure may fail to make all (1,1) matches.[6]

What is surprising is that despite MB being a very weak condition, it is enough to strengthen both of K&M's characterization to any substitutable priority rule.

Theorem 1. *An assignment mechanism ϕ satisfies non-wastefulness, population monotonicity, weak Maskin monotonicity, and MB if and only if $\phi(R, C) = DA(R, C)$ for every substitutable choice rule C and every $R \in \mathcal{R}$.*

Proof. Theorem 2 from K&M establishes that the deferred acceptance algorithm satisfies non-wastefulness, weak Maskin monotonicity, and population monotonicity. The deferred acceptance algorithm outputs a stable assignment. Therefore, it satisfies MB as MB is a necessary condition for stability. We prove the "only if" part using two lemmas.

Lemma 1. *Suppose ϕ satisfies non-wastefulness, population monotonicity, and MB over the set of substitutable priorities. Then $\phi(R, C)$ is stable for*

[5] Both of these mechanisms are discussed in Roth and Sotomayor (1990).

[6] I am grateful to Fuhito Kojima for pointing out this example.

any substitutable priority C.

Proof. For convenience, let $\mu = \phi(R, C)$. Since ϕ is non-wasteful, μ must be individually rational as by definition $\emptyset$ is never scarce. Suppose for contradiction there exists a substitutable priority C and $R \in \mathcal{R}$ such that $\phi(R, C)$ has a blocking pair. Let agent $i \in N$, and object $a \in O$ be such that:

$$aP_i\mu_i \tag{1}$$

$$i \in C_a(\{i\} \cup \mu_a) \tag{2}$$

Since ϕ is non-wasteful, it must be that $|\mu_a| = q_a$. Define R' and R'' as follows:

$$R'_j = \begin{cases} a, \mu_i, \emptyset & j = i \text{ and } \mu_i \neq \emptyset \\ a, \emptyset & j = i \text{ and } \mu_i = \emptyset \\ a, \emptyset & j \in \mu_a \\ R_j & j \notin \mu_a \cup \{i\} \end{cases}$$

$$R''_j = \begin{cases} R'_j & j \in \mu_a \cup \{i\} \\ \emptyset & j \notin \mu_a \cup \{i\} \end{cases}$$

Let $\mu' = \phi(R', C)$. R' is a monotonic transformation of R at μ. Therefore, by weak Maskin monotonicity, $\mu'_j R'_j \mu_j$ for each agent j. In particular, $\mu'_j = a$ for each $j \in \mu_a$. Let $\mu'' = \phi(R'', C)$. By population monotonicity, $\mu'' R''_j \mu'$ for each $j \in \{i\} \cup \mu_a$. This implies for each $j \in \mu_a$, $\mu''_j = \mu'_j = a$. Under R'', the set of agents that find some object acceptable is $\{i\} \cup \mu_a$. Since a is i's top choice under R'' and $i \in C_a(\{i\} \cup \mu_a)$, by MB $\mu''_i = a$. Therefore, $\{i\} \cup \mu_a \subseteq \mu''_a$. Since $|\{i\} \cup \mu_a| = q_a + 1$, this is a contradiction. Therefore, $\phi(R, C)$ is a stable assignment. □

The final step of the proof is to demonstrate that for substitutable priority rules, a stable allocation rule that satisfies weak Maskin monotonicity must be equivalent to the deferred acceptance rule.

Lemma 2. *Consider any substitutable $C \in \mathcal{C}$. If a stable allocation rule ϕ satisfies weak Maskin monotonicity then $\phi(R, C) = DA(R, C)$ for every $R \in \mathcal{R}$.*[7]

Proof. Consider any substitutable $C \in \mathcal{C}$, $R \in \mathcal{R}$, and any stable, weakly Maskin monotonic mechanism ϕ. Let $\mu = DA(R, C)$. As is well known, μ is the agent optimal stable assignment. Let R' denote the truncation of R at μ which is to say if $\mu_i P_i a$ for some school a, then $\emptyset P'_i a$ but otherwise $R_i = R'_i$. It is straightforward to verify that μ is stable under R'. We will demonstrate that μ is the unique stable assignment under R'.

Let μ' denote the agent optimal stable assignment under R' ($\mu' = DA(R', C)$). $\mu' R' \mu$ since μ is stable under R' and μ' is agent optimal stable, but suppose for contradiction there exists an $i \in N$ such that $\mu'_i P'_i \mu_i$. As μ is agent optimal stable under R, μ' is not stable under R. Since $\mu' R \mu$ any blocking pair of μ' under R continues to block μ' under R', a contradiction. Therefore, $\mu = \mu'$. Suppose for contradiction there exists a $\nu \neq \mu$ where ν is stable under R'. As μ is agent optimal under R', $\mu R' \nu$. As ν is individually rational, for every $i \in N$, $\nu_i \in \{\mu_i, \emptyset\}$. Therefore, for every object a, $\nu_a \subset \mu_a$. As $\nu \neq \mu$, there exists an $i \in N$ such that $\nu_i = \emptyset$ but for some object a, $\mu_i = a$. Therefore, $\nu_a \subset \nu_a \cup \{i\} \subset \mu_a$. By the stability of μ, $i \in C_a(\mu_a)$. Therefore, by the substitutability of C, $i \in C_a(\nu_a \cup \{i\})$. Since $a P'_i \emptyset$, i and a block ν, contradicting the stability of ν.

Therefore, μ is the unique stable assignment under R'. As ϕ is a stable mechanism, $\phi(R', C) = \mu$. However, R is a monotonic transformation of R' at μ. Therefore, $\phi(R, C) R \phi(R', C) = \mu$. As μ is agent optimal stable under R and $\phi(R, C)$ is stable, $\phi(R, C) = \mu$. □

[7] Lemma 2 is closely related to but slightly more general than a result in K&M. They demonstrate that for acceptant substitutable priority rules, a stable allocation rule satisfies weak Maskin monotonicity only if it is the deferred acceptance rule. Our proof is a modification of their proof.

Since ϕ is weakly Maskin monotonic by assumption and we have demonstrated that ϕ outputs a stable assignment for all substitutable priority rules, Lemma 2 implies that $\phi(R, C) = DA(R, C)$ for every substitutable choice rule C and every $R \in \mathcal{R}$ $\square$

Theorem 1 limits attention to substitutable priority rules. The next proposition demonstrates that for acceptant[8] priority rules, this is a necessary restriction.

Proposition 1. *Fix an acceptant choice rule C. If an assignment mechanism ϕ satisfies non-wastefulness, population monotonicity, and MB for all $R \in \mathcal{R}$, then C is substitutable.*

Proof. Consider any $i \in N'' \subseteq N' \subseteq N$ and any object a such that $i \in C_a(N')$. Define R' and R'' as follows:

$$R'_j = \begin{cases} a, \emptyset & j \in N' \\ \emptyset & j \in N \setminus N' \end{cases}$$

$$R''_j = \begin{cases} a, \emptyset & j \in N'' \\ \emptyset & j \in N \setminus N'' \end{cases}$$

By MB, $C_a(N') \subseteq \phi_a(R', C)$. Since C is acceptant, $C_a(N') = \phi_a(R', C)$. In particular, since $i \in C_a(N')$, $\phi_i(R', C) = a$. By population monotonicity, $\phi_i(R'', C) = a$. But again, since every agent's top choice is a, by MB and the fact that C is acceptant, $C_a(N'') = \phi_a(R'', C)$. Therefore, $i \in C_a(N'')$ which implies that C is substitutable. $\square$

We next consider K&M's first characterization. The extension of their result to every substitutable priority order follows as an immediate corollary to Theorem 1.

[8] The priority C_a is *acceptant* if for all $N' \subseteq N$, $|C_a(N')| = \min\{|N'|, q_a\}$.

Theorem 2. *An assignment mechanism ϕ satisfies non-wastefulness, IR monotonicity, and MB if and only if $\phi(R, C) = DA(R, C)$ for every substitutable choice rule C and every $R \in \mathcal{R}$.*

Proof. Theorem 2 is an immediate corollary of Theorem 1. IR monotonicity implies both weak Maskin monotonicity and population monotonicity. Therefore, any mechanism that satisfies non-wastefulness, IR monotonicity, and MB also satisfies non-wastefulness, population monotonicity, weak Maskin monotonicity, and MB and is equivalent to the deferred acceptance algorithm for every substitutable priority rule by Theorem 1. K&M prove that the deferred acceptance algorithm satisfies IR monotonicity (their Theorem 1). □

4 Independence

We establish the independence of the axioms in Theorems 1 and 2 through a series of examples. For each example, we assume there are at least two objects a and b, that $|N| \geq 2$, and that $q_a < |N| - 1$. First we demonstrate that mutual best (MB) is independent of non-wastefulness, weak-Maskin monotonicity, population monotonicity, and IR monotonicity. As described in the introduction, the serial dictatorship satisfies all of the axioms in Theorem 1 and 2 except for MB. Example 1 is a different mechanism that satisfies all axioms but MB.

Example 1. (MB Fails) Consider the following mechanism. The mechanism accepts the preferences of the agents and the priorities of the objects. It then disregards the priorities of the objects and runs the deferred-acceptance algorithm as if the priorities of each object are to rank the agents in alphabetical order. As this is an instance of the deferred-acceptance algorithm, it satisfies non-wastefulness, weak-Maskin monotonicity, IR monotonicity, and population monotonicity. However, this mechanism does not satisfy MB.

Given a rank-order list, the top trading cycles algorithm satisfies all of the axioms except for population monotonicity and IR monotonicity.

Example 2. (Population monotonicity fails) Given any priority rule C, if C consists of rank-order lists for each object, then we run top trading cycles. If C_a is not a rank order list for any object a, then we run the deferred acceptance algorithm. Top trading cycles satisfies weak Maskin monotonicity (Papai (2000) and Takamiya (2001)), and it is non-wasteful. However, it is not population monotonic (Abdulkadiroglu and Sonmez (2003)). Since IR monotonicity implies population monotonicity, this mechanism is not IR monotonic. Finally, top trading cycles does satisfy MB.

To establish the independence of non-wastefulness and weak Maskin monotonicity, we modify two examples from K&M.

Example 3. (Wasteful; modified from K&M's Example 2) For any $i \in N$ and $a \in O$, if i lists a first and $i \in C_a(N)$, then assign i to a. Otherwise leave all objects unassigned. This rule is trivially population monotonic, weakly Maskin monotonic, IR monotonic, and satisfies MB. However, it is wasteful.

Example 4. (Not weakly Maskin montonic; modified from K&M's Example 4) We define an allocation rule as follows. For any $i \in N$ and $a \in O$, if i lists a first and $i \in C_a(N)$, then assign i to a. For the remaining objects, fix an agent j. If j reports $R_j^a : a, \emptyset$, then we run a serial dictatorship for the remaining objects and let j pick first. The ordering is otherwise arbitrary. If j reports anything else, then we have j pick last but otherwise keep the same ordering. This rule satisfies MB, is non-wasteful, and is population monotonic. However, it does not satisfy weak Maskin monotonicity. To see this, consider any instance with $|N| - q_a > q_a$ and $j \notin C_a(N)$. Define R as follows:

$$R_i = \begin{cases} b, \emptyset & i \in C_a(N) \\ a, \emptyset & i \notin C_a(N) \end{cases}$$

Let $R'_j = a, b, \emptyset$ and keep $R'_i = R_i$ for $i \neq j$. Then $\phi_j(R, C) = a$. Moreover, R' is a m.t. of R at $\phi(R, C)$; however, $\phi_j(R', C) \neq a$. Therefore, $\phi_j(R, C) R'_j \phi_j(R', C)$ demonstrating that ϕ is not weakly Maskin monotonic.

5 Conclusion

By introducing a fourth axiom, mutual best, we characterize when a mechanism is outcome equivalent to the deferred acceptance algorithm for every substitutable priority rule of the objects being assigned. Mutual best is a significantly weaker assumption than stability. Moreover, it is clearly a desirable property for practical assignment rules.

Our extension sheds further light on the mechanics of the deferred acceptance algorithm and provides an instructive basis for contrasting it with other well studied assignment rules such as top trading cycles and serial dictatorships.

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