Petty Envy When Assigning Objects

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June, 2016

Abstract

Envy of another person's assignment is "justified" if you "deserve" the object and it is possible to assign you to the object. Currently, the literature only considers whether or not the agent deserves the object and ignores whether or not assigning her to it is possible. This paper defines a fair set of assignments in terms of what is possible. We prove that a fair set of assignments has the same properties as the set of stable matches: the Lattice Theorem, Decomposition Lemma, and Rural Hospital Theorem all hold. Moreover, there is a unique, student-optimal fair assignment: the assignment made by Kesten's Efficiency Adjusted Deferred Acceptance mechanism when all students consent.

Consider the problem of assigning agents to objects. A key consideration is what constitutes a fair assignment. Typically, the literature calls an assignment unfair if there is a student i who desires an object a and has higher priority at a than a student j who is assigned to a.[1] But what if it is impossible to assign i to a but it is possible to assign j to a? Ignoring for the moment why it might be possible for one student and not for another, if we honor i's objection, it does not help i

*North Carolina State University. Email address: thayer_morrill@ncsu.edu. This is an early version of the paper "Which School Assignments are Legal?" that was presented at EC '16. Please refer to that paper for a more complete discussion of the topic. I thank Antonio Romero-Medina for the many conversations and espressos. Both were very enjoyable and productive for this work.

[1]This was introduced by Balinski and Sonmez (1999) as the definition of a fair assignment. Following Abdulkadiroglu and Sonmez (2003), the literature typically refers to this as *justified envy*.

but only harms j. Is it fair to not assign j to a just to keep i from being jealous? Typically, people would not call this fair and instead would call it petty.

Consider the following classic example from Roth (1982) and applied to the school assignment problem by Abdulkadiroglu and Sonmez (2003).

Example 1. There are three students i, j, k, and three schools a, b, c, each of which has a capacity of one. The priorities and preferences are as follows:

R_i	R_j	R_k	$\succ_a$	$\succ_b$	$\succ_c$
b	a	a	i	j	j
a	b	b	k	i	i
c	c	c	j	k	k

We start with the assumption that a student with highest priority at an object should either be assigned to that object or an object she prefers. This implies that i must be assigned to b or a. Similarly, j must be assigned to a or b. Therefore, the only two assignments that are possible are the following:

$$\mu = \begin{pmatrix} i & j & k \\ a & b & c \end{pmatrix} \qquad \nu = \begin{pmatrix} i & j & k \\ b & a & c \end{pmatrix}$$

In particular, it is impossible to assign k to either a or b. Consider the assignment ν. k prefers a to her assignment c. k has higher priority at a than the student assigned to a, j. Is k's envy of j "justified"? Our position is no. Since it is impossible to assign k to a, we view this envy as petty.

The key point is that we cannot determine if an assignment is fair in isolation; instead, we must ask if a *set* of possible assignments is fair. A set of assignments determines the assignments which are possible, and the assignments which are possible determines which objections are petty. Central to our concept is requiring a student to block an assignment with an alternative assignment. We say that a student i blocks assignment μ with assignment ν if she prefers her assignment under

ν and she has higher priority at that object then the student assigned to it under μ.[2] For a set of assignments A, we call the set of assignments not blocked by any assignment in A the possible assignments for A and denote it by P. For a set of assignments A to be fair, we require that three conditions are satisfied.

First, we require that no assignment $\mu \in A$ blocks another assignment $\nu \in A$ ($A \subset P$). If so, we would deem the assignment ν "unfair" as a student has justified envy of a school and it's possible to to assign her to that school. Second, we require that no assignment $\mu \in A$ is blocked by an assignment $\nu \in P$. In this case, the objection to μ is valid and not petty. Third, if μ is not blocked by any assignment in P, then we require μ to be included in A. If no assignment in P blocks μ, then either no student has justified envy of μ, or whenever a student has justified envy, it is impossible to assign her to that object. We call a set petty if it does not include such an assignment.

We prove that for any assignment problem there always exists a fair set of assignments. Moreover, we demonstrate that the classical properties that hold for the set of stable marriages in the marriage problem hold for the set of fair assignments. The Lattice Theorem, Decomposition Lemma, and the Rural Hospital Theorem all hold for a fair set of assignments.

The Lattice Theorem implies that there is a well-defined, student-optimal assignment within any fair set of assignments. We demonstrate a surprising result. Any fair set of assignments has the same student-optimal assignment and this assignment is Pareto efficient. Specifically, the assignment made by Kesten's efficiency adjusted deferred acceptance algorithm (Kesten, 2010) is the student-optimal assignment for any fair set of assignments.[3]

Our paper is in the spirit of bargaining sets introduced by Zhou (1994). For a transferable utility, cooperative game, he introduces bargaining sets as a generalization of the core. Specifically, a coalition can block with an alternative only if that alternative is not subsequently blocked. He

[2] For simplicity, in our model we assume that each object has a capacity of one. We extend our results to the general case in the paper "Which School Assignments are Legal?" which is available at http://www4.ncsu.edu/ tsmorril/papers/index.html.

[3] More precisely, the assignment made by the efficiency adjusted deferred acceptance algorithm when all student's consent.

demonstrates that the bargaining set is non-empty for every transferable-utility game. Alcalde and Romero (2015) precedes our paper and is also in the spirit of bargaining sets. Alcalde and Romero (2015) allow a student i to object to school a if there is an assignment with no justified envy in which i is assigned to a.[4] They call unblocked assignments α-equitable and show that an assignment is α-equitable if and only if it weakly Pareto dominates an assignment with no justified envy. While α-equity is similar in spirit to our definition, there are important differences. Most importantly, the set of α-equitable assignments is unfair (using our definition) in the sense that one α-equitable assignment may block another. This can be seen in our Example 2.

1 Model

We assume that there is a set of agents, $A = \{i, j, k, \ldots\}$, to be assigned to a set of objects, $O = \{a, b, c, \ldots\}$. Each agent i has preferences P_i over the objects. Each object a has priorities $\succ_a$ over the agents. For simplicity, we assume that each object may be assigned to only one agent. However, since we are assuming that objects have responsive preferences, all our results continue to hold when an object a may be assigned to $q_a > 1$ many agents.

An assignment μ is a function from agents to objects such that no two agents are assigned to the same object. $\mu_i = a$ indicates that agent i is assigned to object a. In a slight abuse of notation, $\mu_a = i$ means that i is the agent who received a as an assignment. $\mu_i = \emptyset$ indicates that agent i is left unassigned.

[4] This is not the way α-equity is defined in Alcalde and Romero (2015), but it is equivalent to their definition. Specifically, they define (i, μ') to be an ϵ-objection to μ if $\mu'_i P_i \mu_i$ and $i \succ_{\mu'_i} j$ where $\mu_j = \mu'_i$. An ϵ-objection (i, μ') is admissible if no student has an ϵ-objection to μ'. An assignment μ is α-equitable if there is no admissible ϵ-objection to μ. An ϵ-objection is equivalent to an assignment being blocked by another assignment. Since any assignment with justified envy can be blocked, only an assignment with no justified envy has no counter objections. Therefore, an assignment is α-equitable if it is not blocked by any assignment that eliminates justified envy. Reasonable stability, introduced by Cantala and Papai (2014), is closely related to α-equity.

2 Results

An assignment μ **blocks** an assignment ν if there is an agent i such that $\mu_i = aP_i\nu_i$ and $i \succ_a \nu_a$. Given a set of assignments S, μ is a **possible** assignment for S if μ is not blocked by any assignment in S. For convenience, given a set of assignments A, we let $\pi(A)$ denote the set of possible assignments (assignments that are not blocked by an assignment in A).

Definition 1. F is a **fair set of assignments** if:

1. No assignment in F is blocked by another assignment in F (each assignment in F is possible: $F \subset \pi(F)$)
2. No assignment in F is blocked by a possible assignment for F ($F \subset \pi(\pi(F))$), and
3. If μ is not blocked by any possible assignment, then $\mu \in F$ ($\pi(\pi(F)) \subset F$).

If condition 3 is violated, we will say the set of assignments is **petty**.

An assignment that eliminates justified envy is not blocked by any assignment. Therefore, assignments that eliminate justified envy are included in any fair set of assignments.

There is a natural connection between our notion of fairness and von Neumann Morgenstern stable sets (hereafter vNM stable). A set of assignments S is **vNM stable** if $S \subset \pi(S)$ (internal stability) and $\pi(S) \subset S$ (external stability). Equivalently, S is vNM stable if and only if $S\pi(S)$. Conditions (2) and (3) imply that for a fair set F, $\pi(\pi(F)) = F$. This is a weaker condition than vNM stability as if $\pi(F) = F$ then $\pi(\pi(F)) = F$. At least theoretically, there could be a cycle of assignments such as the following: μ blocks ν which blocks ψ which blocks μ. If this were the case, then a fair set of assignments wouldn't exist. $\{\mu\}$ is not fair because it doesn't include ψ which is unblocked. However, $\{\mu, \psi\}$ is unfair because ψ blocks μ. But $\{\psi\}$ is unfair as it does not include ν which is unblocked, and so on.[5]

[5]We thank a referee for pointing out the connection between vNM stability and fairness. There are several papers regarding vNM stability for the marriage problem that have a close relationship to the current paper. Please

Theorem 1. *For any preferences P and priorities $\succ$, a fair set of assignments always exists.*

Proof. The following fact will be useful:

Monotonicity Fact: If $A \subseteq B$, then $\pi(B) \subseteq \pi(A)$.

Let S^1 be the set of assignments with no justified envy. S^1 may be "too small" in the sense that it may be petty. Let $B^1 = \pi(S^1)$. These are the assignments that are possible for S^1. B^1 may be "too big" in the sense that one possible assignment may block another possible assignment. Let $S^2 = \pi(B^1)$. We will show that $S^1 \subseteq S^2 \subset B^1$. Iterating this procedure, we will show that $S^1 \subseteq \ldots \subseteq S^{n+1} \subseteq B^n \subseteq \ldots B^1$. In particular, there must be an n such that $S^n = S^{n+1}$. We will show that such a steady state is a fair set of assignments.

We first show that $S^1 \subseteq S^2 \subseteq B^1$. Consider any $\mu \in S^1$. μ has no justified envy, so μ is not blocked by any assignment. $B^1 = \pi(S^1)$ and $S^2 = \pi(B^1)$. Since μ is not blocked by any assignment, $\mu \in B^1$ and $\mu \in S^2$. Therefore, $S^1 \subseteq B^1$ and $S^1 \subseteq S^2$. Since $S^1 \subseteq B^1$; $B^1 = \pi(S^1)$; and $S^2 = \pi(B^1)$, by the monotonicity fact, $S^2 \subseteq B^1$. So indeed, $S^1 \subseteq S^2 \subseteq B^1$.

In general, let B^k be the assignments that are possible for S^k, let S^{k+1} be the assignments that are possible for B^k. Our inductive hypothesis is that $S^{n-1} \subseteq S^n \subseteq B^{n-1}$ and we show that $S^n \subseteq S^{n+1} \subseteq B^n \subseteq B^{n-1}$. Ours is a telescoping sequence, and the result follows by repeatedly applying the monotonicity fact.

1. Since $S^{n-1} \subseteq S^n$, $\pi(S^n) = B^n \subseteq \pi(S^{n-1}) = B^{n-1}$: $B^n \subseteq B^{n-1}$.
2. Since $B^n \subseteq B^{n-1}$, $\pi(B^{n-1}) = S^n \subseteq \pi(B^n) = S^{n+1}$: $S^n \subseteq S^{n+1}$.
3. Since $S^n \subseteq B^{n-1}$ (by the inductive hypothesis), $\pi(B^{n-1}) = S^n \subseteq \pi(S^n) = B^n$: $S^n \subseteq B^n$.
4. Since $S^n \subseteq B^n$, $\pi(B^n) = S^{n+1} \subseteq \pi(S^n) = B^n$: $S^{n+1} \subseteq B^n$.

see the extended version of the current paper, titled "Which School Assignments are Legal?" and available at http://www4.ncsu.edu/ tsmorril/papers/index.html, for a discussion on the relationship and differences between the current paper and Ehlers (2007), Wako (2008), Wako (2010), and Bando (2014).

This proves the desired result. For all n, $S^1 \subseteq \ldots S^n \subseteq S^{n+1} \subseteq B^n \subseteq \ldots B^1$. Since these are finite sets, there must be an n such that $S^n = S^{n+1}$. We claim that S^n is fair. By definition, the set of possible assignments for S^n is B^n. S^{n+1} are the assignments not blocked by a possible assignment. Since $S^n = S^{n+1}$, no assignment in S^n is blocked by a possible assignment. Moreover, if an assignment μ is not blocked by a possible assignment, then $\mu \in S^{n+1}$ and therefore $\mu \in S^n$.

$\square$

We now fix a set of fair assignments F, and we say an assignment μ is fair if $\mu \in F$. To avoid confusion, we will call an assignment that eliminates justified envy a **stable** assignment. An important result from classical matching theory is that the set of stable assignments has an elegant structure. In particular, the set of stable assignments is a lattice (the Lattice Theorem), and if an agent is unassigned in one stable match, she is unassigned in all stable matches (the Rural Hospital Theorem). We now demonstrate that a fair set of assignments has the same structural properties as the set of stable assignments.

Given fair assignments μ and ν, define $I(\mu) = \{i \in I | \mu_i P_i \nu_i\}$ and $S(\mu) = \{a | \mu_a \succ_a \nu_a\}$ (with $I(\nu)$ and $S(\nu)$ defined analogously).

Lemma 1. *(Decomposition Lemma) Given two fair assignments μ and ν, μ and ν map $I(\mu)$ onto $S(\nu)$ and $I(\nu)$ onto $S(\mu)$.*

Proof. Suppose $i \in I(\mu)$ and let $a = \mu_i$. Since $i \in I(\mu)$, $\mu_i P_i \nu_i$. If $i \succ_a \nu_a$, then μ would block ν. Since ν is fair and therefore not blocked by another fair assignment, it must be that $\nu_a \succ_a i = \mu_a$. Since, $\mu_i \in S(\nu)$, μ maps $I(\mu)$ into $S(\nu)$. An immediate consequence of this is that $|I(\mu)| \leq |S(\nu)|$. Analogously, consider $a \in S(\nu)$. Since $a \in S(\nu)$, $\nu_a = i \succ_a \mu_a$. If $\nu_i = a P_i \mu_i$, then ν would block μ. Therefore, $\mu_i P_i \nu_i$, and, ν maps $S(\nu)$ into $I(\mu)$. Consequently, $|S(\nu)| \leq |I(\mu)|$. This implies that $|I(\mu)| = |S(\nu)|$. The analogous condition for $I(\nu)$ and $S(\mu)$ follows by symmetry. $\square$

Given two assignments μ and ν, define $\mu \vee \nu_i = \max_i \{\mu_i, \nu_i\}$. Define $\mu \wedge \nu_i = \min_i \{\mu_i, \nu_i\}$

Theorem 2. *(Lattice Theorem) Let μ and ν be fair assignments. Then $\mu \vee \nu$ and $\mu \wedge \nu$ are fair assignments.*

Proof. Let $\lambda = \mu \vee \nu$ and let $\tau = \mu \wedge \nu$. The Decomposition Lemma establishes that λ and τ are proper assignments. Suppose for contradiction that a student i blocks λ with a possible assignment λ': $\lambda'_i = aP_i\lambda_i$ and $i \succ_a \lambda_a$. Let $\lambda_a = j$, and without loss of generality, $\mu_j = a$. Since $\lambda_i R_i \mu_i$ by definition and $\lambda'_i P_i \lambda_i$, $\lambda'_i P_i \mu_i$. Since $\lambda'_i = aP_i\mu_i$ and $i \succ_a \mu_a$, λ' blocks μ, a contradiction since μ is fair. Similarly, suppose i is blocks τ with a possible assignment τ'. Let $a = \tau'_i$. Without loss of generality, $\tau_i = \mu_i \neq a$. By the Decomposition Lemma, $\tau_a = \max_a \{\mu_a, \nu_a\}$. Since $i \succ_a \tau_a$, $i \succ_a \mu_a$. Since τ' is possible, $\tau'_i = aP_i\mu_i$, and $i \succ_a \mu_a$, it follows that τ' blocks μ. Therefore, μ is not fair, a contradiction. $\square$

There are a number of interesting conclusions that are an immediate consequence of the Lattice Theorem. We now know that any fair set of assignments contains an agent optimal fair assignment. Moreover, we can conclude that there is at most one Pareto efficient assignment in any fair set of assignments. Otherwise, if μ and ν are both Pareto efficient, and contained in a fair set of assignments, then $\mu \vee \nu$ would be a well defined assignment that Pareto dominated both. Later, we will demonstrate that in any fair set of assignments, there always exists a Pareto efficient assignment.

Corollary 3. *In any fair set of assignments, there exists at most one Pareto efficient assignment.*

Proof. Suppose for contradiction that μ and ν are both Pareto efficient and both contained in a fair set of assignments. By the Lattice Theorem, $\mu \vee \nu$ is a well defined assignment. But since μ and ν cannot be Pareto ranked, $\mu \neq \mu \vee \nu \neq \nu$. Therefore, $\mu \vee \nu$ Pareto dominates both μ and ν which is a contradiction as both assignments are Pareto efficient. $\square$

Theorem 4. *(Rural Hospital Theorem) For any student i, if i is unassigned in one fair assignment, then i is unassigned in every fair assignment.*

Proof. Given two assignments μ and ν, let $I(\mu)$ and $S(\mu)$ be defined as in the Decomposition Lemma. Let μ^I be the agent optimal fair assignment, and let μ be any other fair assignment. Let $J = \{i|\mu_i^I \neq \mu_i\}$. Since μ^I is agent-optimal, $J = I(\mu^I)$. For any $i \in J$, $\mu_i^I \neq \emptyset$ since μ is individually rational and $\mu_i^I P_i \mu_i$. Moreover, by the Decomposition Lemma, μ is an onto map from $S(\mu)$ to $I(\mu^I)$. Therefore, there exists an object $a \in O$ such that $\mu_a^I = i$. Therefore, $\mu_i = a$, and i is assigned to an object in every fair assignment. □

3 Relationship to the Efficiency Adjusted Deferred Acceptance Algorithm

Kesten (2010) introduces a new mechanism for the school assignment problem: the Efficiency Adjusted Deferred Acceptance Algorithm (hereafter EADAM). Kesten identifies the source of DA's inefficiency, called interrupter students, and resolves this inefficiency by introducing EADAM. For the precise formulation of the mechanism, we refer the reader to Kesten (2010). EADAM is a subtle and complicated mechanism, but Kesten proves that 1) a student is never harmed by consenting to having her priority waived; 2) EADAM Pareto dominates DA; and 3) if all students consent, then EADAM is Pareto-efficient.[6]

In this section, we prove a surprising result. Any fair set of assignments contains the assignment made by EADAM when all students consent. Since the EADAM assignment is Pareto efficient and a fair set of assignments is a lattice, this proves that EADAM Pareto dominates any other fair assignment. This characterization is analogous to Gale and Shapley's (1962) characterization of DA: the DA assignment is each man's favorite stable assignment. Our result demonstrates that the

[6]Two recent papers have established some of the properties of EADAM. Dur and Morrill (2016) demonstrates that EADAM only Pareto improves DA when students submit the same preferences. When students are strategic, they provide an example in which at least one student is made worse off relative to DA in every Nash equilibrium. A corollary of this result is that, in equilibrium, a student may be harmed by consenting. Dur, Gitmez, and Yilmaz (2015) provides the first characterization of EADAM. They prove that it is the unique mechanism that is partially fair, constrained efficient, and gives each student the incentive to consent.

EADAM assignment is each student's favorite fair assignment.

We will use the simplified EADAM mechanism (hereafter sEADA) introduced by Tang and Yu (2014). The key part of sEADA is the concept of an underdemanded school. For a given assignment μ, a school a is *underdemanded* if for every student i, $\mu_i R_i a$. There are several facts about underdemanded schools that critical for Tang and Yu's mechanism. First, under the DA assignment, there is always an underdemanded school. For example, the last school that any student applies to is an underdemanded school. Second, a student assigned by DA to an underdemanded school cannot be part of a Pareto improvement. Using these facts, Tang and Yu define sEADA iteratively as follows:[7]

The simplified Efficiency Adjusted Deferred Acceptance Mechanism (sEADA)

Round 0: Run DA on the full population. For each student i assigned to an underdemanded school a, assign i to a; remove i; and reduce a's capacity by one.

Round k: Run DA on the remaining population. For each student i assigned to an underdemanded school a, assign i to a; remove i; and reduce a's capacity by one.

Tang and Yu (2014) prove that sEADA and Kesten's EADAM make the same assignment. We will prove that the sEADA assignment is in any fair set of assignments and therefore Pareto dominates any other fair assignment. The following example provides the intuition for this result.

Example 2.

R_i	R_j	R_k	R_l	$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
b	a	a	b	i	j	k	l
a	c	c	d	k	l		
	b			j	i		

In Round 0 of sEADA, the DA assignment is:

$$\begin{pmatrix} i & j & k & l \\ a & b & c & d \end{pmatrix}$$

[7] To be precise, this is the definition of sEADA when all students consent to allowing their priority to be violated.

Note that i envies j's assignment. j envies i and k's assignment, and k envies i's assignment. However, no student strictly prefers d to her assignment. Therefore, d is an underdemanded school. We assign l to d and remove both the agent and the object. Now the assignment problem is:

R_i	R_j	R_k	$\succ_a$	$\succ_b$	$\succ_c$
b	a	a	i	j	k
a	c	c	k	i	
	b		j		

The DA assignment of this problem (and therefore the Round 1 sEADA assignment) is:

$$\begin{pmatrix} i & j & k \\ b & c & a \end{pmatrix}$$

Student l has justified envy of the Round 1 sEADA assignment. Specifically, bP_ld and $l \succ_b i$. However, l cannot be part of a Pareto improvement of the Round 0 assignment since she is assigned to an underdemanded school. The consequence is that in any assignment where l is assigned to b, a student is made worse off relative to the Round 0 assignment. One can verify that this student blocks the new assignment with the DA assignment. Therefore, l has justified envy of b, but it is impossible to assign her to b. Therefore, l's objection is petty, and consequently, the Round 1 assignment is fair as it is not blocked by any possible assignment.

In the Round 1 assignment, the underdemanded schools are b and c. After removing i and j, all that remains is k and a; therefore, the assignment problem is trivial. The Round 2 sEADA assignment is to assign k to a. Therefore, the final sEADA assignment is:[8]

[8] Note that this example also demonstrates that the set of α-equitable assignments, as defined by Alcalde and Romero (2015), is unfair. There are two Pareto improvements of the DA assignment:

$$\mu = \begin{pmatrix} i & j & k & l \\ b & a & c & d \end{pmatrix} \qquad \nu = \begin{pmatrix} i & j & k & l \\ b & c & a & d \end{pmatrix}$$

Both μ and ν are α-equitable since they both Pareto improve the DA assignment. ν blocks μ (k prefers ν and has higher priority at a than does j); therefore, the set of α-equitable assignments is not fair.

$$\begin{pmatrix} i & j & k & l \\ b & c & a & d \end{pmatrix}$$

There is a natural relationship between whether or not an assignment is possible and whether or not a student is assigned to an underdemanded school. No student has justified envy of the DA assignment. Suppose student i's DA assignment is a but she strictly prefers school b which is assigned by DA to student j. Consider an alternative assignment ν where i is assigned to b and j is assigned to a school she likes less than b. The DA assignment is included in any fair set of assignments. Therefore, ν is not a possible assignment since j blocks ν with the DA assignment. Specifically, an assignment where a student receives a school she prefers to her DA assignment is only unblocked if it is a Pareto improvement of the DA assignment. We emphasize the following point as it is essential for our analysis.

Fact: (Lemma 1, Tang and Yu 2014) At the DA matching, no student matched with an underdemanded school is Pareto improvable.

Each iteration of sEADA Pareto improves the DA assignment. The only students with justified envy of this assignment have been removed in an earlier round. Students are only removed if they are assigned to an underdemanded school, and students who are assigned to an underdemanded school are not part of any Pareto improvements of the DA assignment. Therefore, although an underdemanded student may have justified envy of school a, she is not assigned to a in any possible assignment. Therefore, her objection is petty, and she is not able to block the Pareto improvement of DA.

Lemma 2. *Let $\mu = DA(P)$ and suppose μ_i is underdemanded. Then for any assignment ν such that $\nu_i P_i \mu_i$, ν is blocked by μ. In other words, if i's DA assignment is underdemanded, then it is not possible to assign i to a school she prefers to her DA assignment. Any objection i makes to such a school's assignment is petty.*

Proof. The key point is that essentially underdemanded students cannot be part of a Pareto improvement to μ. Let ν be any assignment such that $\nu_i = a$. Define a cycle C as follows. Let $i_1 = i$ and let $a_1 = \nu_{i_1} = a$. In general, let $i_k = \mu_{a_{k-1}}$ and $a_k = \nu_{i_k}$. Since there are only a finite number of agents, eventually $i_n = i_1$. i_1 prefers $\nu_{i_1} = a$ to μ_{i_1}. If every agent in the cycle preferred ν to μ, then reassigning each student in the cycle to her assignment under ν would Pareto improve μ. Since i is essentially underdemanded and therefore cannot be part of a Pareto improvement, there is at least one student in the cycle who prefers her assignment under μ to her assignment under ν. Let i_m be the first student that prefers $a_{m-1} = \mu_{i_m}$ to $a_m = \nu_{i_m}$. In particular, i_{m-1} prefers $\nu_{i_m - 1} = a_{m-1}$ to $\mu_{i_{m-1}}$. μ has no justified envy, and i_{m-1} prefers a_{m-1} to her assignment under μ. Therefore, she has lower priority at a_{m-1} than i_m does. In summary, μ is a fair assignment. $\mu_{i_m} = a_{m-1} P_{i_m} \nu_{i_m}$ and $i_m \succ_{a_{m-1}} \nu_{a_{m-1}}$. Therefore, agent i_m blocks ν with μ, and therefore, ν is not a possible assignment. $\square$

Theorem 5. *The EADA assignment is a fair assignment.*

Proof. We proceed by induction. Set $P^1 = P$, and let μ^1 be $DA(P)$. We define preferences P^k inductively as follows. Let $\mu_i^{k-1} = DA(P^{k-1})$. Let U^{k-1} be the essentially underdemanded students. Define P^k as follows. If $i \in U^{k-1}$, then move μ_i^{k-1} to the top of i's preferences; otherwise, leave i's preferences unchanged. If $i \notin U^{k-1}$, then set $P^k = P^{k-1}$. We prove by induction that for every $n \geq 1$, μ^n is fair. First, consider the base step. Only an underdemanded student could have justified envy of μ^2. But if student i is underdemanded and i envies school a, Lemma 2 establishes that a is not possible for i. Therefore, μ^2 is not blocked by any possible assignment, and consequently, μ^2 is fair. Our inductive hypothesis is to assume for each $k < n$ that (a) μ^k is fair and (b) if i is underdemanded at μ^k and $aP_i\mu_i$, then a is impossible for i.

First, we prove that μ^n is fair under P. Only students that were underdemanded in a previous round have had their preferences changed. No student has justified envy of μ^n under preferences P^n, therefore, if i has justified envy of μ^n under the true preferences P, then i was underdemanded in some previous round $k < n$. By the inductive hypothesis, any object i prefers to $\mu^k = \mu^n$ is not possible for i. Therefore, μ^n is not blocked by any possible assignment, and it therefore fair.

Second, we prove that if μ_i^n is essentially underdemanded and $aP_i\mu$, then i is not assigned to a under any possible assignment.

By Lemma 2, a is not possible under P^n. In particular, let ν be any assignment such that $\nu_i = a$. ν is blocked by some student j and by μ^n under P^n. If $P_j^n = P_j$, the j blocks ν under P. If $P_j^n \neq P_j^n$, then j was essentially underdemanded in some round k. If $\nu_j P_j \mu_j^n$, then by the inductive hypothesis, ν_j is not possible for j and therefore ν is blocked by some student. If $\mu_j^n R_j \nu_j$, then since j blocks ν under P_j^n, j still blocks ν with μ_j^n under P_j.

Since each μ^n equals the n^{th} round assignment in sEADA, and sEADA's final assignment equals the EADA assignment, the EADA assignment is fair. $\square$

4 Conclusion

When a school board chooses an assignment mechanism, it typically balances strategyproofness, efficiency, and fairness. While the definitions of strategyproofness and efficiency are objective, what constitutes a fair assignment is subjective. Typically, the literature has considered an assignment unfair if a student wants and deserves an alternative assignment. We propose instead that an assignment is unfair if a student wants and deserves an alternative assignment, and it is possible to assign her to that school. Strengthening the definition of what is unfair in this way leads to a starkly different conclusion as to what is the most efficient fair assignment. In particular, by taking into consideration what is possible, Kesten's EADAM assignment is fair and Pareto dominates any other fair assignment.

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