The Impossibility of Restricting Tradeable Priorities in School Assignment

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Abstract

Top Trading Cycles is an efficient school assignment mechanism that respects each school's priorities up to the school's capacity; however, it is impossible for any efficient mechanism to respect all of each school's priorities. We address whether or not it is possible to respect any intermediate level of priorities. Unfortunately, we prove that no efficient mechanism is always able to respect more priorities than the number of students it has capacity for. Our motivation is whether or not it is possible for a school board to designate certain priorities as untradeable (such as sibling or walk-zone priorities). We model this formally and demonstrate that it is not possible without severe unintended consequences.

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1 Introduction

The central tension in school assignment is that there does not always exist an assignment which is Pareto efficient and fair in the sense of respecting student preferences and priorities. A number of papers have considered how to make a fair assignment more

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efficient, how to make efficient assignments fairer, and whether or not alternative fairness concepts are compatible with efficiency.[1] Our paper seeks to quantify the extent to which an efficient assignment mechanism is able to respect priorities.

The Top Trading Cycles mechanism (hereafter TTC) has the desirable property of *respecting top priorities*. Specifically, for a school a with capacity q, if a student has one of the q highest priorities at a, then she is guaranteed to be assigned to a or a school she strictly prefers. In this sense, although TTC does not respect all of a school's priorities, it does respect the priorities up to a school's capacity. The question we address is whether any efficient mechanism respects more than these top priorities.

Unfortunately, we find a negative result. We demonstrate that it is not possible for an efficient mechanism to respect any additional top priorities.[2] Therefore, TTC is optimal among efficient mechanisms in the sense that no efficient mechanism respects more priorities than does TTC.

We further show that even much weaker notions of efficiency are incompatible with protecting more priorities when student strategies are taken into consideration. An assignment is defined to be perfect if every student is assigned to her favorite school.[3] Clearly a perfect assignment is not always possible, but we prove that a mechanism that respects more than top priorities and a version of Maskin monotonicity, which we call top-move invariance[4], does not always make a perfect assignment even when one exists. We prove that this result extends even if we only desire constrained efficiency. Moreover, these results hold if we replace top-move invariance with a weak consistency notion.[5] It is intriguing that these basic properties are incompatible.

Our question is motivated by the following practical question. Boston was the first school district to choose between a fair and an efficient mechanism for school assignment. In explaining the reasoning for not choosing TTC, Superintendent Payzant writes:[6]

[1] See Kesten (2010), Morrill (2014), Dur, Gitmez, and Yilmaz (2015), Morrill (2016), and Kloosterman and Troyan (2016).

[2] Specifically, we mean that no efficient mechanism always respects additional priorities. Of course, it is possible for an efficient mechanism to respect more priorities in some problems.

[3] To the best of our knowledge, the definition of perfect was introduced by Aziz, Brandt, and Harrenstein (2013).

[4] A mechanism is top-move invariance if a student cannot change its outcome by moving her assignment at the top of her submitted preferences.

[5] A mechanism is weakly consistent if the removal of an unassigned student does not change the assignments of the other students.

[6] This memo can be found at http://www.iipsc.org/resources/tpayzant-memo-05.25.2005.pdf. Accessed August 6, 2013.

> Another algorithm we have considered, Top Trading Cycles Mechanism, presents the opportunity for the priority for one student at a given school to be "traded" for the priority of a student at another school, assuming each student has listed the others school as a higher choice than the one to which he/she would have been assigned. There may be advantages to this approach, particularly if two lesser choices can be traded for two higher choices. It may be argued, however, that certain priorities e.g., sibling priority apply only to students for particular schools and should not be traded away.

Superintendent Payzant's concern regarding TTC raises the following design question. Is it possible to design a trading mechanism where students are allowed to trade some priorities but not others? To answer this question we consider a general framework of trading mechanisms in the spirit of Papai's hierarchical exchange mechanisms Pápai (2000). We say a mechanism is a trading mechanism if it is an iterative process that proceeds as follows. Each spot at a school is allocated to some student (where the spots may be allocated to just one student or a variety of students). In each round, the mechanism determines a trade between students (where it is possible that the trade is trivial in the sense of a student taking one of her own spots). We say such a trading mechanism respects restricted priorities if the mechanism never selects a trade involving a restricted priority such as sibling priority. We show that it is not possible to design a trading mechanism with basic fairness and efficiency properties without creating severe unintended consequences. Specifically, a student may be made worse off by having the highest priority at a school (if the priority is restricted) then if instead she had the lowest priority.

Our paper contributes to the growing literature on the efficient assignment of students to indivisible schools when no school is owned by any of the students. This topic was pioneered by Pápai (2000) who introduced the version of TTC that we study here. TTC is part of a broader class of mechanisms introduced by Pápai (2000) called hierarchical exchange rules. Pycia and Ünver (2011a) introduce a class of trading mechanisms called trading cycles that extend hierarchical exchange rules. Kesten (2004) introduced an alternative trading algorithm called Equitable Top Trading Cycles. Morrill (2014) introduces an alternative called Clinch and Trade. Both Equitable Top Trading Cycles and Clinch and Trade are designed to make an efficient assignment with fewer instances of justified envy than TTC.

This problem is important because of its applicability to assigning students to public schools. This problem was first considered by Balinski and Sönmez (1999) and then by Abdulkadiroğlu and Sönmez (2003). Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discusses the market design considerations of applying DA and TTC to the school assignment problem.

The organization of the rest of the paper is as follows: in the next section we describe the model and the axioms we use in our analysis. In Section 3 we demonstrate the impossibility results. In Section 4, we introduce a class of trading mechanisms and show that students might be punished when their priorities are improved.

2 Model

We define a school choice problem as a list $(I, S, q, P, \succ, e)$ where

- I is the set of students,
- S is the set of schools,
- $q = (q_s)_{s \in S}$ is the quota vector where q_s is the number of available seats at school s,
- $P = (P_i)_{i \in I}$ is the preference profile where P_i is the strict preference of student i over the schools and the option of being unassigned which we denote by $s_\emptyset$,
- $\succ = (\succ_s)_{s \in S}$ is the priority profile where $\succ_s$ is the strict priority relation of school s over I,
- $e = (e_s)_{s \in S}$ is the protected (restricted) priority profile where e_s is the number of students with a protected priority for school s.

Each parameter is standard with the exception of the protected priorities. Protected priorities are the key innovation of this paper and will be explained in greater detail later. We set $q_{s_\emptyset} = |I|$ since we assume no restriction on the number of students who may be unassigned. We will often refer to the **rank** of a student at a particular school. The rank of student i at school $s \in S$, denoted by $r_s(i)$, is defined as $r_s(i) := |\{j | j \succeq_s i\}|$. Let R_i be the at-least-as-good-as relation associated with P_i for all $i \in I$. We say a school s is acceptable for student i if $s P_i s_\emptyset$ and school s is unacceptable for student i if $s_\emptyset P_i s$.

Given a school choice problem, an **assignment** $\mu : I \to S \cup \{s_\emptyset\}$ is a function which assigns each student to a school, possibly $s_\emptyset$, such that the number of students assigned to a school is less than or equal to its capacity. For a given assignment μ, the assignment of student i is denoted by μ_i, and the set of students assigned to s is denoted by μ_s. Let μ_{-s} be the set of students not assigned to school s, i.e., $\mu_{-s} = I \setminus \mu_s$.

An assignment μ is **perfect** if all students in I are assigned to their most preferred school in $S \cup \{s_\emptyset\}$. Clearly, in general a perfect assignment need not exist. Moreover, for any problem there exists at most one perfect assignment.

The following definitions are standard in the school choice literature. We include them for completeness of the paper. An assignment μ **Pareto dominates** another assignment ν if each student $i \in I$ weakly prefers her assignment under μ to her assignment under ν and there exists at least one student j who strictly prefers her assignment under μ to her assignment under ν. An assignment μ is **Pareto efficient** if there does not exist another assignment ν which Pareto dominates μ. For brevity, we will say efficient instead of Pareto efficient.

An assignment μ is **individually rational** if each student $i \in I$ is not assigned to an unacceptable school, i.e., $\mu_i \ R_i \ s_\emptyset$ for all $i \in I$. An assignment μ is **nonwasteful** if there does not exist a student school pair (i, s) such that $sP_i\mu_i$ and $|\mu_s| < q_s$. It is straightforward to verify that if an assignment is wasteful or individually irrational, then it is not Pareto efficient. Similarly, if an assignment is Pareto inefficient, then it is not perfect. An assignment μ is **fair** if there does not exist a student school pair (i, s) where $s \ P_i \ \mu_i$ and $i \succ_s j$ for some $j \in \mu_s$. A weaker fairness and nonwastefulness notion is mutual best. An assignment μ satisfies **mutually best** if there does not exist a student-school pair (i, s) such that s is i's favorite school, i is the highest-ranked at s, but i and s are not assigned.

Given (I, S, q, e), a profile of students, schools, capacities, and protected priorities, a **mechanism** ϕ is a function which maps preferences and priorities to an assignment. The assignment under ϕ for the preference and priority profile $(P, \succ)$ is denoted by $\phi^{[I,S,q,e]}(P, \succ)$ and i's assignment under ϕ is denoted by $\phi_i^{[I,S,q,e]}(P, \succ)$. When (I, S, q, e) are clear from context, we will denote $\phi^{[I,S,q,e]}(P, \succ)$ by $\phi(P, \succ)$. In the rest of the paper, we refer quadruple (I, S, q, e) as **subproblem**.

For each property of an assignment, we say that a mechanism ϕ satisfies that property if for all school choice problems $(I, S, q, P, \succ, e)$, the assignment $\phi^{[I,S,q,e]}(P, \succ)$ satisfies the property. For example, a mechanism ϕ is nonwasteful if for all assignment problems $(I, S, q, P, \succ, e)$, the assignment $\phi^{[I,S,q,e]}(P, \succ)$ is nonwasteful. A mechanism ϕ is perfect if it selects the perfect assignment whenever it exists.

A mechanism ϕ is **strategy-proof** if for any assignment problem $(I, S, q, P, \succ, e)$ there does not exist a student i and a preference relation P_i' such that $\phi_i(P', P_{-i}, \succ) \; P_i \; \phi_i(P_i, \succ)$. A mechanism ϕ is **nonbossy** if a student cannot change the assignment of the other students without changing her own assignment by submitting different preference list. That is, ϕ is **nonbossy** if for any P and P_i' $\phi_i(P_i', P_{-i}, \succ) = \phi_i(P, \succ)$ implies $\phi(P_i', P_{-i}, \succ) = \phi(P, \succ)$.

We will use a weak version of Maskin (1999) monotonicity.[7] Given a student i and an assignment μ, P_i is μ_i**-on-top** if $\mu_i P_i s$ for every school $s \in ((S \cup \{s_\emptyset\}) \setminus \{\mu_i\})$. We define a mechanism ϕ to be **top-move invariant** if for all preferences P and all students i, if P_i' is $\phi_i(P, \succ)$-on-top, then $\phi(P, \succ) = \phi(P_i', P_{-i}, \succ)$.

It is not obvious that top-move invariance relates directly to nonbossiness. Top-move invariance says that a particular way of changing preferences does not change any student's assignment. Nonbossiness says that if one students change does not affect her own assignment, then it does not affect any other student's assignment. However, as the next result shows, top-move invariance is a stronger condition than nonbossiness (Proposition 1). When we restrict our attention to strategy-proof mechanisms, the two conditions are equivalent (Proposition 2).

Proposition 1 *Any top-move invariant mechanism is nonbossy.*

Proof. Suppose ϕ is top-move-invariant, and consider any student i and any preferences P_i, P_i', and P_{-i} such that $\phi_i(P, \succ) = \phi_i(P', P_{-i}, \succ) = s$. Let $\bar{P}_i$ be any preferences that ranks s first. Since ϕ is top-move invariant $\phi(\bar{P}, P_{-i}, \succ) = \phi(P, \succ)$ and $\phi(\bar{P}, P_{-i}, \succ) = \phi(P', P_{-i}, \succ)$. Therefore, $\phi(\bar{P}, P_{-i}, \succ) = \phi(P) = \phi(P', P_{-i}, \succ)$ which implies that ϕ is nonbossy. ■

It is straightforward to verify that top-move invariance is a strictly stronger condition than non-bossiness. For example, consider the following variation of a serial dictatorship. In a standard serial dictatorship, when it is a student's turn, she chooses her favorite school. Consider instead the mechanism which assigns the dictator her least favorite school. This mechanism is nonbossy, but when the initial dictator ranks her assignment

[7]We will not use Maskin monotonicity in our analysis, but we define it here for completeness. R_i' is a **monotonic transformation** of R_i at $s \in S$ if any school ranked above s under R_i' is also ranked above s under R_i (that is, $s' R_i' s \Rightarrow s' R_i s$ for every school s'). R' is a monotonic transformation of R at assignment μ if for every student i, R_i' is a monotonic transformation of R_i at μ_i. A mechanism is **Maskin monotonic** if whenever R' is a monotonic transformation of R at μ, then $\phi(R') = \phi(R)$.

first, she no longer receives it. Therefore, it is not top-move invariant. However, the next result demonstrates that among strategy-proof mechanisms, the two conditions are equivalent.

Proposition 2 *Any strategy-proof and nonbossy mechanism is top-move invariant.*

Proof. Let ϕ be strategy-proof and nonbossy. Consider a problem P. Suppose under $\phi(P, \succ)$, i is assigned to a and let P_i' be any preference in which i ranks a at the top of the list. If $\phi_i(P_i', P_{-i}, \succ) \neq a$, then $\phi_i(P, \succ) P_i' \phi_i(P_i', P_{-i}, \succ)$ which would violate strategy-proofness. Therefore, $\phi_i(P_i', P_{-i}, \succ) = a$. Since ϕ is nonbossy and $\phi_i(P, \succ) = \phi_i(P_i', P_{-i}, \succ)$, $\phi(P, \succ) = \phi(P_i', P_{-i}, \succ)$. Therefore, ϕ is top-move invariant. ■

Pápai (2000) proved that TTC is strategy-proof and nonbossy. Therefore, Proposition 2 implies that TTC is top-move invariant. It is worth mentioning that the school proposing DA is top-move invariant and nonbossy but is not strategy-proof.[8]

3 Results

The fundamental tension in school choice is that there does not in general exist an assignment which is fair and Pareto efficient. TTC has the property that it respects the top q_s priorities at each school s. However, a natural question is whether or not it is ever possible to respect any additional priorities.

Unfortunately, we prove here that it is never possible. Specifically, we consider the following weakening of fairness and prove that it is incompatible with any Pareto efficient mechanism. For notational convenience, we define $\bar{0}$ and $e^{\max}$ to be the $|S|$-dimensional vectors where each entry is 0 and $|I|$, respectively.

Definition 1 *Let $\bar{0} \leq e \leq e^{\max}$. An assignment μ is **e-fair** if there does not exist a student i and a school s such that (i) $sP_i\mu_i$; (ii) $r_s(i) \leq e_s$; and (iii) $i \succ_s j$ where $\mu_j = s$. A mechanism $\phi^{(I,S,q,e)}$ is e-fair if for every $(P, \succ)$, $\phi^{(I,S,q,e)}(P, \succ)$ is e-fair.*

[8] Let s be student i's assignment under the school proposing DA. No school s' that i strictly prefers to s proposed to i. Therefore, whether or not i ranks this school above s is irrelevant to the outcome of school proposing DA. Hence, the school proposing DA is top-move invariant. For the nonbosiness we refer to Afacan and Dur (2017).

Similar to the definition of stability in Section 2, we say an assignment μ is e-**stable** if it is e-fair, nonwasteful and individually rational. A mechanism $\phi^{(I,S,q,e)}$ is e-stable if for every $(P, \succ)$, $\phi^{(I,S,q,e)}(P, \succ)$ is e-stable.

Condition (ii) is the novel part of Definition 1. In particular, if $e = e^{\max}$, then e-fairness and fairness are equivalent. Since there is no fair and efficient mechanism, there is no mechanism that is efficient and $e^{\max}$-fair. As noted, if for every school s, $e_s \leq q_s$, then TTC is Pareto efficient and e-fair. Our main result is to demonstrate that for any nontrivial assignment problem and any intermediate e, there does not exist a Pareto efficient and e-fair mechanism. Since e-fairness is a weaker condition than fairness, this result is stronger than the incompatibility between Pareto efficiency and fairness (Balinski and Sönmez, 1999). We define a problem to be nontrivial if there are at least two schools, $|S| \geq 2$, and for any two schools $a, b \in S$, $q_a + q_b \leq |I|$ (i.e. that there is potentially competition for each school). These two conditions are very natural when we consider school choice in practice.

Proposition 3 *Let (I, S, q, e) be a non-trivial subproblem and let $0 \leq e \leq e^{\max}$. Then if $e_s \geq q_s$ for every school s and $e_s > q_s$ for some school s, then no mechanism is Pareto efficient and e-fair.*

Proof. Fix a school a such that $e_a > q_a$. Let b be any other school in S. Label a subset of the students $I_1 = \{i_1, i_2, \ldots, i_{q_a-1}\}$, $I_2 = \{j_1, j_2, \ldots, j_{q_b-1}\}$, and $\{i, j, k\}$. This requires there to be at least $q_a + q_b + 1$ students which follows by our nontriviality assumption ($q_a + q_b < |I|$). Let $\succ_a$ rank students in $\{i_1, i_2, \ldots, i_{q_a-1}, i, k, j\}$ as the $q_a + 2$ highest-ranked students, in that order. Let $\succ_b$ rank students in $\{j_1, j_2, \ldots, j_{q_b-1}, j, i, k\}$ as the $q_b + 2$ highest-ranked students, in that order. Rank the remaining students at a, b and all students for all other schools arbitrarily.

Define the students preferences as follows (where $1 \leq m \leq q_a - 1$ and $1 \leq n \leq q_b - 1$):

P_{i_m}	P_{j_n}	P_i	P_j	P_k
a	b	b	a	a
$s_\emptyset$	$s_\emptyset$	a	b	$s_\emptyset$
		$s_\emptyset$	$s_\emptyset$	

Suppose ϕ is Pareto efficient and e-fair. Let μ be the assignment made by ϕ under $(P, \succ)$. Hence, μ is e-fair. If μ is wasteful, then μ cannot be Pareto efficient. Hence, we take μ as nonwasteful. First, $\mu_{i_m} = a$ for any student i_m where $0 \leq m \leq q_a - 1$ since by construction each student i_m has one of the q_a highest priorities. Note that if $\mu_j = a$, then $\mu_k \neq a$. Since k has the $q_a + 1$ highest priority at a, $k \succ_a j$, $e_a \geq q_a + 1$, and μ is e-fair, this would be a contradiction. Therefore, $\mu_j \neq a$. Since j has one of the q_b highest priorities at b (and $e_b \geq q_b$) and j is not assigned to a, her first choice, j must be assigned to b, her second choice. Similar to above, μ_{j_n} must be assigned to b for any $0 \leq n \leq q_b - 1$. Since j and $\{j_1, j_2, \ldots, j_{q_b - 1}\}$ are assigned to b, i is not assigned to b. Therefore, i is assigned to a as a is i's second favorite school and she has one of the q_a highest priorities at a. However, this implies that μ is not Pareto efficient as reassigning i to b and j to a is a Pareto improvement. ■

We will show that this result is quite general by considering alternative conditions. For a mechanism, perfection is a significantly weaker requirement than Pareto efficiency. For example, the student proposing DA mechanism is perfect but Pareto inefficient. However, when a mechanism is top-move invariant (nonbossy), even this level of efficiency is incompatible with e-fairness (e-stability).

Proposition 4 *Let (I, S, q, e) be a non-trivial subproblem and let $0 \leq e \leq e^{\max}$. Then if $e_s \geq q_s$ for every school s and $e_s > q_s$ for some school s, then no perfect mechanism is*

1. *top-move invariant and e-fair; or*

2. *nonbossy and e-stable.*

Proof. We prove (1) and (2) by using parallel arguments. Suppose ϕ satisfies the axioms either in (1) or (2). We consider the example used in the proof of Proposition 3. Suppose each $i' \in (I \setminus (I_1 \cup I_2 \cup \{i, j, k\}))$ finds each school in S unacceptable, i.e. $s_\emptyset P_{i'} s$ for each $s \in S$. We denote this problem with $(P, \succ)$.

Under $\phi(P, \succ)$, if a student $i' \notin (I_1 \cup \{i, k\})$ is assigned to a, then by e-fairness each student in $(I_1 \cup \{i, k\})$ is assigned to an acceptable school for her. Similarly, under $\phi(P, \succ)$, if a student $i' \notin (I_2 \cup \{j\})$ is assigned to b, then by e-fairness each student in $(I_2 \cup \{j\})$ is assigned to an acceptable school for her. Hence, by feasibility k cannot be assigned to a or b, and i and j cannot be assigned to their top choices under $\phi(P, \succ)$. Moreover, if ϕ is e-stable, then k is assigned to $s_\emptyset$.

Let P'_k be a preference order in which $\phi_k(P, \succ)$ is ranked as the first choice. If ϕ is top-move invariant, then by definition $\phi(P, \succ) = \phi(P'_k, P_{-k}, \succ)$. If ϕ is e-stable, then $\phi_k(P'_k, P_{-k}, \succ) = \phi_k(P, \succ) = s_\emptyset$ and nonbossiness implies $\phi(P, \succ) = \phi(P'_k, P_{-k}, \succ)$. However, under $(P'_k, P_{-k}, \succ)$ there exists a perfect assignment and ϕ fails to select it. Hence, ϕ is not perfect, a contradiction. ■

One can wonder whether there exists a mechanism satisfying all axioms mentioned in both parts of Proposition 4 except one. The answer is affirmative. The student proposing DA mechanism is perfect and e-stable (as it is stable). The TTC mechanism is perfect and top-move invariant. The school proposing DA is top-move invariant and e-stable.[9]

These results demonstrate that there are significant costs to protecting more priorities than the capacity of a school. Intuitively, any attempt to limit the trading of some priorities may limit the trading of any priorities. We will model this explicitly in the next section. Next, we consider another way of weakening Pareto efficiency. We show that it is not even possible to achieve e-fairness so that the outcome is undominated by any other e-fair assignment. We call this efficient e-fairness.

Definition 2 *Let $\bar{0} \leq e \leq e^{\max}$. An assignment μ is **efficiently e-fair** if it is e-fair and there does not exist an alternative assignment ν which is e-fair and Pareto dominates μ. A mechanism $\phi^{(I,S,q,e)}$ is efficiently e-fair, where $\bar{0} \leq e \leq e^{\max}$ if for every $(P, \succ)$, $\phi^{(I,S,q,e)}(P, \succ)$ is efficiently e-fair.*

We define efficient e-stability for an assignment and mechanism analogous to Definition 2. We first illustrate through a simple example that the student proposing DA mechanism is not efficiently e-fair.

Example 1 *Let $I = \{i, j, k\}$, $S = \{a, b\}$, and $q_s = e_s = 1$ for all $s \in S$. The preferences and priorities are given as:*

P_i	P_j	P_k	$\succ_a$	$\succ_b$
a	b	a	j	i
b	a	$s_\emptyset$	k	j
$s_\emptyset$	$s_\emptyset$		i	k

Under this problem, student proposing DA mechanism selects assignment μ where $\mu_i = b$, $\mu_j = a$ and $\mu_k = s_\emptyset$. However, there exists an e-fair assignment ν which Pareto dominates μ: $\nu_i = a$, $\nu_j = b$ and $\nu_k = s_\emptyset$.

One can wonder whether there exists an efficiently e-fair mechanism. Recall that, for a given problem if assignment μ is e-fair, then for any $s \in S$ priorities of top $q_s + e_s$ students are respected for school s. In a recent paper Dur, Gitmez, and Yilmaz

[9] Recall that e-stability implies e-fairness and top-move invariance implies nonbossiness.

(2015) show that an efficiently e-fair assignment always exists.[10] Despite this positive observation, in the following Proposition we show that efficient e-fairness and top-move invariance are incompatible.

Proposition 5 *Let (I, S, q, e) be a non-trivial subproblem and let $0 \leq e \leq e^{\max}$. Then if $e_s \geq q_s$ for every school s and $e_s > q_s$ for some school s, then no mechanism is*

1. *top-move invariant and efficiently e-fair; or*

2. *nonbossy and efficiently e-stable.*

Proof. We prove (1) and (2) by using parallel arguments. Suppose ϕ satisfies the axioms either in (1) or (2). We use the same problem used in the proof of Propositions 3 and 4.

In problem $(P, \succ)$, there exists a unique efficiently e-fair assignment, denoted by μ, in which all students in $I_1 \cup \{i\}$ are assigned to a, all students in $I_2 \cup \{j\}$ are assigned to b, and all the other students are assigned to $s_\emptyset$. That is, $\phi(P, \succ) = \mu$. Note that under μ, i and j are assigned to their second choices. Moreover, if ϕ is e-stable, then $\phi_k(P, \succ) = s_\emptyset$.

Let P'_k be a preference order in which $\phi_k(P, \succ)$ is ranked as the first choice. If ϕ is top-move invariant, then by definition $\phi(P, \succ) = \phi(P'_k, P_{-k}, \succ)$. If ϕ is e-stable, then $\phi_k(P'_k, P_{-k}, \succ) = \phi_k(P, \succ) = s_\emptyset$ and nonbossiness implies $\phi(P, \succ) = \phi(P'_k, P_{-k}, \succ)$. However, under $(P'_k, P_{-k}, \succ)$, there exists e-stable, therefore e-fair, assignment in which all students are assigned to their top choices. Hence, ϕ is not efficiently e-fair, therefore e-stable, a contradiction. ■

Next we consider the strategic implications of protecting priorities and again we find a negative result.

Proposition 6 *Let (I, S, q, e) be a non-trivial subproblem and let $0 \leq e \leq e^{\max}$. Then if $e_s \geq q_s$ for every school s and $e_s > q_s$ for some school s, then no mechanism is nonbossy, strategy-proof, and e-stable.*

Proof. Suppose ϕ satisfies all of these axioms. By the definition of e-stability, ϕ is nonwasteful, individually rational, and e-fair. We use the same problem used in the proof of Propositions 3 and 4.

[10] They do not use the term e-fair, but they show that when only some priorities need to be respected, it is always possible to find a Pareto undominated assignment that Pareto improves the student-optimal stable assignment and respects the protected priorities. Such an assignment is e-fair.

As explained in the proof of Proposition 4, due to e-fairness and feasibility, k cannot be assigned to a or b, and i and j cannot be assigned to their top choices under $\phi(P, \succ)$. Moreover, by individual rationality, $\phi_k(P, \succ) = s_\emptyset$.

Let P'_k be a preference order in which $\phi_k(P, \succ) = s_\emptyset$ is ranked as the first choice. By Proposition 2, ϕ is top-move invariant. Hence, $\phi(P, \succ) = \phi(P'_k, P_{-k}, \succ)$.

Consider the preference order P'_i in which b is the only acceptable school. In problem $(P'_i, P'_k, P_{-\{i,k\}}, \succ)$ due to strategy-proofness and individual rationality i will be assigned to $s_\emptyset$, $\phi_i(P'_i, P'_k, P_{-\{i,k\}}, \succ) = s_\emptyset$. Moreover, due to nonwastefulness and e-fairness all students in I_1 and I_2 are assigned to a and b under $\phi(P'_i, P'_k, P_{-\{i,k\}}, \succ)$, respectively. Then, due to nonwastefulness and e-fairness, j is either assigned to a or b under $\phi(P'_i, P'_k, P_{-\{i,k\}}, \succ)$. Due to individual rationality, all the other students are assigned to $s_\emptyset$.

If $\phi_j(P'_i, P'_k, P_{-\{i,k\}}, \succ) = b$, then the available seat in a is wasted, i.e. $a \; P_j \; \phi_j(P'_i, P'_k, P_{-\{i,k\}}, \succ)$ and $|\phi_a(P'_i, P'_k, P_{-\{i,k\}}, \succ)| < q_a$. If $\phi_j(P'_i, P'_k, P_{-\{i,k\}}, \succ) = a$, then the available seat in b is wasted, i.e. $b \; P_j \; \phi_j(P'_i, P'_k, P_{-\{i,k\}}, \succ)$ and $|\phi_b(P'_i, P'_k, P_{-\{i,k\}}, \succ)| < q_b$. This contradicts with the fact that ϕ is nonwasteful. ■

As in Proposition 4, there exists a mechanism satisfying all axioms in Proposition 6 except one. TTC mechanism is nonbossy and strategy-proof. Student proposing DA mechanism is strategy-proof and e-stable. School proposing DA mechanism is nonbossy and e-stable.

It is worth mentioning that, Propositions 2 and 6 imply that top-move invariance and e-stability are incompatible.

Pápai (2000) shows that combination of strategy-proofness and non-bossiness is equivalent to group strategy-proofness. Moreover, Pápai (2000) demonstrates that a mechanism is group-strategy-proof, Pareto efficient, and reallocation proof if and only if it is a hierarchical exchange rule. Hence, as a direct corollary of Proposition 6, any hierarchical exchange rule fails to be e-fair when $e_s \geq q_s$ for every school s and $e_s > q_s$ for some school s.

Corollary 1 *Let (I, S, q, e) be a non-trivial subproblem and let $0 \leq e \leq e^{\max}$. Then if $e_s \geq q_s$ for every school s and $e_s > q_s$ for some school s, then there does not exist a hierarchical exchange rule which is e-fair.*

We also introduce an alternative axiom which is closely related to consistency. A mechanism is said to be consistent if whenever a set of students and their assignment are removed, then the assignment of the remaining students does not change. The axiom we consider is much weaker because we only impose a restriction when unassigned students are removed.

In particular, a mechanism $\phi^{(I,S,q,e)}$ is **weakly consistent** if for every $(P,\succ)$ and every student i such that $\phi_i^{(I,S,q,e)}(P,\succ)=s_\emptyset$, then $\phi_j^{(I\setminus\{i\},S,q,e)}(P_{-i},\succ)=\phi_j^{(I,S,q,e)}(P,\succ)$ for all $j\in I\setminus\{i\}$. For instance, TTC mechanism is not consistent but it satisfies weak consistency. In the following proposition, we state that the impossibility results above hold when we include weak consistency.

Proposition 7 *Let (I,S,q,e) be a non-trivial subproblem and let $0\leq e\leq e^{\max}$. Then if $e_s\geq q_s$ for every school s and $e_s>q_s$ for some school s, then no mechanism is:*

- *weakly consistent, perfect, and e-stable.*
- *weakly consistent and efficiently e-stable.*
- *weakly consistent, strategy-proof, and e-stable.*

Proof. We refer to the proofs of Propositions 4, 5, and 6. ■

4 Trading Mechanisms

In this section we focus on a class of trading mechanisms which includes TTC (Abdulkadiroğlu and Sönmez, 2003; Pápai, 2000). In particular, we investigate whether there exists a trading mechanism which does not allow protected priorities to be traded. Before starting our analysis we define the class of trading mechanisms.

We say a mechanism ϕ belongs to the class of trading mechanisms if the followings are true:

1. For any problem $(P,\succ)$, ϕ selects cycles[11] recursively and in each cycle $(i_1,s_1,i_2,s_2,...,i_n,s_n)$ each student i_x is assigned to s_x for all $x\in\{1,...,n\}$ where $i_{n+1}=i_1$. Denote the first cycle selected in problem $(P,\succ)$ by $\phi^c(P,\succ)$.

2. The (first) cycle selected in a problem $(P,\succ)$ is not affected by the preference profile of the students who are not in the cycle, i.e., $\phi^c(P,\succ)=\phi^c(P_{I^c},\tilde{P}_{-I^c},\succ)$ for any possible $\tilde{P}_{-I^c}$ where I^c is the set of students in $\phi^c(P,\succ)$.

A cycle $(i_1,s_1,i_2,s_2,...,i_n,s_n)$ **respects the protected priorities** if there does not exist $x\in\{1,2,...,n\}$ such that $r_{s_x}(i_x)>q_{s_x}$ and i_{x+1} has protected priority for s_x.

[11] A cycle $(i_1,s_1,i_2,s_2,...,i_n,s_n)$ is a list of students and schools in which student i_x points to s_x and s_x points to i_{x+1} for all $x\in\{1,...,n\}$ where $i_{n+1}=i_1$.

A trading mechanism ϕ respects the protected priorities if for any problem $(P, \succ)$ the (first) cycle selected by ϕ, $\phi^c(P, \succ)$, respects the protected priorities.

We now demonstrate that an attempt to limit what priorities are tradeable (transferable) may lead to severe unintended consequences under trading mechanisms. We show that if a school does not allow some of the priorities to be traded (transferred), then a student can be harmed by having a protected priority. For example, if we do not allow students to trade sibling priorities, then a student may prefer to be ranked last by a school then to have the highest, but protected sibling priority. For convenience, we present the argument as a comparison between having the highest, but protected priority versus the lowest, but unprotected priority. However, the same argument implies that a student can be better off by being declared unacceptable by a school than by having the highest priority at the same school.[12] We consider this an unintended consequence because surely the intention of restricting a priority such as sibling priority is not to harm the students that have an older sibling attending a school they are not interested in attending.

Proposition 8 *Let ϕ be any mutually best and nonwasteful trading mechanism which respects the protected priorities. Under ϕ, a student can be worse off having the highest priority at a school if this priority is protected than having the lowest (but unprotected) priority.*

Proof. Consider the following problem: $S = \{a, b\}$, $I = \{i, j\}$ and $q = (1, 1)$. The priorities and preferences are given as:

$\succ_a$	$\succ_b$	P_i	P_j
i	j	b	a
j	i	a	b
		$s_\emptyset$	$s_\emptyset$

Suppose only i has protected priority at a, i.e., $e_a = 1$ and $e_b = 0$. Suppose ϕ is a trading mechanism and satisfies all axioms stated above. Since both students find both schools acceptable, in any nonwasteful assignment seats at a and b are filled. Then, we consider following possible cycles which can be selected by ϕ firstly:

[12] We do not frame it this way since we have not allowed schools to declare students unacceptable in our model. However, all algorithms could be easily modified to allow this and the same result would hold.

1. $i \to a \to i$,
2. $i \to b \to i$,
3. $j \to a \to j$,
4. $j \to b \to j$,
5. $i \to b \to j \to a \to i$,
6. $i \to a \to j \to b \to i$.

Note that, under (1), (4), and (6) ϕ assigns i to a and j to b. While under (2), (3), and (5) ϕ assigns i to b and j to a.

By assumption, i has protected priority at a. Therefore, ϕ cannot select cycle (5). Otherwise, ϕ fails to satisfy respecting the protected priorities. Suppose ϕ selects (2). Then, due to mutual best j can change the assignment of i who is in the first selected cycle by ranking b as first choice. This is also true for cycle (3). Hence, ϕ assigns i to a.

Now consider the following priority structure in which i has the lowest, but unprotected priority at a.

a	b	P_i	P_j
j	j	b	a
i	i	a	b
		$s_\emptyset$	$s_\emptyset$

We can interpret this priority profile as follows: In the first problem i has sibling priority but in the second one i does not have sibling priority.

In this second problem any mutually best mechanism assigns j to a. And in order not to waste the seat in b, i will be assigned to b: i becomes better off by losing her sibling priority at school a. ■

An alternative way of presenting Proposition 8 is to use a similar concept to the respecting improvement in the test scores introduced by Balinski and Sönmez (1999). Since the priority structure in the school choice environment does not only depend on the test scores we use an axiom called respecting improvement in the priorities. We say that $\tilde{\succ}$ is an improvement in the priorities for student $i \in I$ if:

(1) $i \succ_s j \Longrightarrow i \tilde{\succ}_s j$ for all $s \in S$,

(2) there exists at least one student j and school s' such that $j \succ_{s'} i \tilde{\succ}_{s'} j$, and

(3) $k \succ_s l \Longleftrightarrow k \tilde{\succ}_s l$ for all $s \in S$ and $l, k \in I \backslash \{i\}$.

A mechanism ϕ **respects improvements in the priorities** if $\tilde{\succ}$ is an improvement in the priorities for student $i \in I$, then $\phi_i(P, \tilde{\succ}) R_i \phi_i(P, \succ)$. That is, a mechanism respects improvements in the priorities if a student is not punished for having higher priorities for some schools. Now we can re-state our result in Proposition 8 by using respecting improvements in the priorities.

Corollary 2 *There does not exist a mutually best, nonwasteful trading mechanism which respects the protected priorities and improvements in the priorities.*

Proof. We refer to the proof of Proposition 8. ■

5 Conclusion

This paper explores the extent to which respecting priorities is compatible with efficiency in school assignment. Unfortunately, we demonstrate a general negative result: it is not possible to respect more priorities than the capacity of a school without conflicting with even the most basic efficiency properties. The key implication of this result is whether or not it is possible for a school board to allow some priorities to be traded but not school-specific priorities such as sibling attendance or being within walking distance to the school. This paper demonstrates that it is impossible to design a trading mechanism that makes a subset of the priorities untradeable without sacrificing even the most basic efficiency properties. This suggests that if the trading of some priorities is completely unacceptable to a school board, then they should use the DA mechanism instead of the TTC mechanism.

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