Family Ties: School Assignment With Siblings

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Abstract

We introduce a new school choice problem motivated by the following observations: students are assigned to *grades* within schools, many students have siblings who are applying as well, and many school districts require siblings to attend the same school. The latter disqualifies the standard approach of considering grades independently as it separates siblings. We argue that the central criterion in school choice—elimination of justified envy—is now inappropriate, as it does not consider siblings. We propose a new solution concept that addresses this. Moreover, we introduce a *strategy-proof* mechanism that satisfies it. Using data from the Wake County magnet school assignment, we demonstrate the impact on families of our proposed mechanism versus the "naive" assignment wherein sibling constraints are not taken into account. Interestingly, the problem can be equivalently modeled within the many-to-many matching with contracts framework, and our results have novel implications in this literature. Despite the fact that *neither* families' nor schools' choice functions are substitutable (even bilaterally), we show that there always exists a *stable* assignment.

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1 Introduction

School assignment is now one of the most well-studied topics in market design. In this expansive literature, every paper that we are aware of considers the problem of assigning students to schools. In practice, however, a student is assigned to a *grade* within a school. It is tempting to dismiss this observation as a trivial technicality. Indeed, Gale and Shapley (1962) showed that a model where schools have multiple seats is essentially the same as the marriage market, lending credence to the idea that we may consider each grade independently without loss of generality. There is a simple reason why this typically is not possible: school boards often require that siblings attend the same school, and many children have siblings involved in the same assignment procedure. It is immediately clear that an assignment protocol that considers grades independently may separate a large number of siblings.

We encountered this problem when designing and administering the assignment procedure for the Wake County Public School System's (WCPSS) magnet program. In the 2017-2018 academic year, WCPSS was the 15th largest school system in the United States with a total of 160,429 students. The district is comprised of 183 schools (grades K-12) with over 38 magnet schools; seats at the latter are assigned to students via a school choice program.[1] Over 20% of students in the system have siblings in the K-12 grades. It is common for multiple siblings to apply for a magnet assignment at the same time; about 13% of students fall into this category. This occurs, for example, when a family has just moved to Wake County or when a family decides to try for a (different) magnet program after the older child has already begun attending school. Since student assignments across grades are determined simultaneously, it is not possible to automatically give the highest priority to siblings. These situations are likely to persist, as Wake County is one of the fastest growing counties in the country and a significant proportion of the growth is attributed to migration.[2]

Both WCPSS and parents view sibling separation as unacceptable. WCPSS states

[1] See Dur, Hammond, and Morrill (2018), and Dur, Hammond, and Kesten (2018) for additional details about Wake County.

[2] On average, more than 40 people moved to Wake County per day in 2017, accounting for 68% of the growth. See http://www.wakegov.com/census.

explicitly that assigning siblings together is one of its strictest requirements:

> "The highest priority in any of the application processes is for entering grade siblings to attend the same school as an older sibling, so long as the siblings live at the same address... This means that if you apply for more than one sibling to attend a school, the application process will not select one sibling without the other. If there are not available seats for each sibling, the program will select none of the siblings." -WCPSS[3]

Similarly, on a message board created by WCPSS to elicit feedback on policy changes, parents voiced strong concerns about separating siblings:

> "...it is NOT RIGHT for students or to ask parents to decide between sacrificing what's good for one child... in order to have children at the same school versus keeping one child... and other sibling(s) who forcibly must attend a different school. The latter situation is easily and potentially a logistical nightmare." -Parent in WCPSS Online Discussion[4]

If we assign students to grades independently, then many siblings would be assigned to different schools. Alternative methods are required, then, to avoid misassignment. Hence, we are faced with a real-life problem affecting hundreds of families with school-aged children. These complications are likely for any school district that has a policy of keeping siblings together.

This motivates a new market design problem: assigning families to schools. We generalize the standard school choice problem of Abdulkadiroğlu and Sönmez (2003) by specifying which students are siblings and splitting schools to grades. In order to be considered for her corresponding grade, each student still reports a ranking over schools, but (as WCPSS policy dictates) siblings are required to report the same preference ranking over schools. Furthermore, since WCPSS "will not select one sibling without the other", we constrain feasible assignments as such.

[3]Retrieved June 2018 at https://www.wcpss.net/Page/33755. Also see the Wake County Board of Education Policy Manual, Policy 6200 and 6200 R & P "Student Assignment".

[4]Retrieved August 2018 at https://wcpss.granicusideas.com/discussions/2018-19-enrollment-proposal-draft-1/topics/feedback-about-proposed-grandfathering-changes.

The central criterion in the school choice literature is respecting a student's priority.[5] We say that student i has justified envy at, or "blocks", an assignment if i prefers a school s to her own assignment, and i has a higher priority than a student assigned s. An assignment with justified envy is typically interpreted as being unfair.[6]

This definition is no longer appropriate when there are siblings. We discuss two reasons why. First, if i has a sibling, then it is not enough for i to have justified envy at a school for the assignment to be unfair. For i to attend that school, her sibling must also have a sufficiently high priority at that school. A relevant notion of justified envy must somehow take into account all of i's siblings; the standard definition does not. Second, suppose i has justified envy of j, i does not have siblings, but j does. In this case, removing j would also remove all of j's siblings (even if the latter have high priority). This could result in empty seats.[7] While school boards may have different ways of weighing capacity utilization versus honoring priorities, at WCPSS (and the other school systems we have spoken to) the clear primary objective is to maximize the number of children who are able to participate in the magnet program, while respecting priorities is secondary.

We generalize what constitutes a block to resolve these issues. Intuitively, under no justified envy, a student ranked x^{th} can block the assignment of the y^{th} ranked-student when $x < y$ and they are in the same grade. We extend this to allow the x^{th} and w^{th} ranked students to block the y^{th} and z^{th} ranked students so long as $x < y$ and $w < z$, and they are in the same respective grades. Specifically, we allow a group of students J to block an assignment if there is a group of students K such that the students of J, one by one, have justified envy over the students of K.[8] Both J and K must be "closed under siblings" in the sense that if J (K) contains one sibling, then it must contain all siblings. Otherwise, replacing J with K would possibly separate some set of siblings and the resulting assignment would not be feasible.

Our definition of blocking generalizes the standard definition of justified envy since when

[5] Many school districts including, Boston, New York, and Chicago, have adopted an assignment procedure that respects students' priorities.

[6] See also the student placement problem of Balinski and Sönmez (1999).

[7] This would violate the priority of j's sibling, as she has justified envy of the "empty seat".

[8] Our formal definition allows for J to have more students than K if the school in question is not assigned to its full capacity.

there are no siblings, the two notions are equivalent. A coalition can be a mixture of siblings, only children, and students from more than one grade. Thus, our blocking coalitions can be quite complex and allow for far more general combinatorial patterns of blocking than the simple blocking considered under justified envy. Note that a group of students need not be related to block others. We define an assignment as *suitable* if there is no such blocking coalition.[9]

Does a *suitable* assignment always exist? There is reason to be pessimistic. Our problem is closely related to the famous matching with couples problem in the context of matching doctors and hospitals; in this problem, a *stable* assignment—a closely related equilibrium notion—might not exist. Intuitively, this is because some doctors (those that are single) view the hospitals as substitutes while other doctors (those that are couples) view the hospitals as complements. We will show that in our problem not only do sibling preferences violate substitutability, but any choice function used by the schools will also violate substitutability. We give a more detailed discussion of similarities and differences in the Related Literature section.

Our main result is to show that despite this, a *suitable* assignment always exists (Theorem 2). We do this by introducing a new family of mechanisms called Sequential Deferred Acceptance (SDA) whose members always select a *suitable* assignment. Furthermore, we show that no student can manipulate these mechanisms by reporting false preferences—each mechanism is *strategy-proof*, or dominant-strategy incentive compatible (Theorem 2).

Using data from the WCPSS magnet program assignment for the 2018-2019 school year, we compare the assignment of our new mechanism versus the "naive" grade-by-grade Deferred Acceptance assignment. The latter generates 196 instances of sibling mismatch; the SDA brings this figure down almost to 0. Does this mean that the SDA simply ignores individual priorities for the sake of keeping siblings together? It turns out, this concern does not materialize. The SDA is able to keep siblings together at a very low "cost": only 17 students in the entire Wake County school district were 1) assigned to a school (because their sibling was) and 2) violated another student's priority.

Our results are also of interest in the matching with contracts literature initiated by Hatfield and Milgrom (2005). Their generalization of the two-sided matching problem

[9]Neither suitability nor group stability implies each other.

allows agents not just to match but to match with certain terms or conditions—the set of possibilities defines the set of possible contracts. In their "many-to-one" model of workers and firms, each worker has a preference over the contracts (involving them), and each firm has a choice function defined over the sets of contracts (involving them). A set of contracts (an assignment) is *stable* if no agent wants to drop any currently held contract, and no other set of contracts would be chosen by some agents in lieu of the current ones. Hatfield and Milgrom (2005) showed that substitutability of the firms' choice functions was sufficient to guarantee the existence of a *stable* assignment. As such, the property (and subsequent relaxations) is a central point of discussion in this literature. If in addition, the Law of Aggregate Demand is satisfied, then there is a *strategy-proof* mechanism that selects a *stable* assignment. We further discuss this topic in the Related Literature section.

School assignment with siblings fits naturally within the "many-to-many" variant of the matching with contracts framework. Yenmez (2018) studies the college admissions problem with early decisions, and is the only other paper that we are aware of that also examines an application of this variant. Our problem is many-to-many because a family possibly needs multiple seats at a school, and each seat at a grade may be thought of as a contract between the parents and the school. Our results tell a surprising and contrasting story here: Despite the fact that neither schools nor families have substitutable choice functions, a *stable* assignment always exists. Furthermore, schools' choice functions do not satisfy the Law of Aggregate Demand, yet there is still a mechanism that selects a *stable* assignment.

1.1 Related Literature

We contribute to the literature on school choice pioneered by Balinski and Sönmez (2003) and Abdulkadiroğlu and Sönmez (2003). We briefly mention some key contributions in school choice: Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2005) examine school choice programs in Boston and NYC; Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak, and Roth (2009) study issues regarding priority classes and the breaking of ties; and Kesten (2010) proposes a new mechanism for improving on the efficiency of the student optimal stable match. To the best of our knowledge, we are the first to explicitly model siblings, grades, and the ensuing institutional constraint that siblings must be assigned to the same school. Several recent papers also study extensions where students may apply as groups or are assigned sequences of seats over time: Dur and Wiseman (2018) consider a problem where students may express preferences to attend the same school as their neighbors, as opposed to siblings. Kennes, Monte, and Tumennasan (2014) study the Danish daycare assignment system where each child is assigned to a school each period, so that each student must express preferences over sequences of schools.[10]

Our paper is related to matching with couples in the context of assigning doctors to hospitals. In these papers, couples are allowed to apply together and submit rankings over *pairs* of (possibly different) hospitals. Roth (1984) showed that when couples are present, there may be no *stable* assignment. Klaus and Klijn (2005) provide maximal domain results (in terms of allowable preferences for the couples) to ensure the existence of a *stable* assignment. Some also consider the case (as in ours) where each couple must always be assigned to the same hospital (see Dutta and Masso (1997)), but have different assumptions about hospital preferences.[11]

The key theoretical difference between our paper and the matching with couples literature is that sibling contraints give rise to particular types of complementarities exhibited uniformly by all sibling pairs and schools.[12] These types are not present in matching with couples (as there are no "grades"). In some ways, this makes the problem more difficult: all schools and many agents have complementarities in their preferences. In other ways, the problem is easier: since the complementarities are based on grades and sibling contraints, there is a common structure across schools' preferences.

Roth and Peranson (1999) provide a detailed overview of the algorithms implemented in the US National Resident Matching Program, wherein couples may apply to hospitals together. Kojima, Pathak, and Roth (2013) similarly study the matching market for psychologists (run by the Association of Psychology Postdoctoral and Internship Centers), and propose a theoretical framework for modeling large markets. Empirically, the couples problem is significantly different from the siblings problem. Table 1 collates key average statistics

[10]Kurino (2014) studies a related dynamic problem where instead of each school having a priority, each agent's current assignment is treated as an endowment.

[11]For example, our school's choice functions do not satisfy Dutta and Masso's weaker form of substitutability (group substitutability). See example in the proof of Theorem 1.

[12]Section 2.2 provides a formal discussion.

Table 1: Matching With Couples vs. School Assignment With Siblings

	NRMP (1993-1996)	Psychology (1999-2007)	WCPSS (2018)
Couples / Siblings as % of Applicants	4.1%	1.3%	20%
Number Hospitals / Schools	3,755	1,094	38
Average Capacity	6.1	2.5	30

from the NRMP, the market for psychology internships, and magnet school admission in Wake County.[13] Couples make up a substantially smaller proportion of the total applicants than sibling pairs. A large number of hospitals have low capacity, while a small number of magnet programs have high capacity. Roth and Peranson (1999) found that surprisingly despite the negative theoretical results, a *stable* assignment existed in each year they observed (for the NRMP). Kojima, Pathak, and Roth (2013) show that when there are few couples and preference lists are short (relative to the whole market size), the probability of a stable assignment approaches one as the market grows. Using several years of data from the market for clinical psychologists, they also find a *stable* assignment in each year. In extreme contrast, for each of the four years in which we have run the WCPSS school assignment, there has never been an assignment without justified envy and keeps siblings together.

Our problem can be modeled in the matching with contracts framework initiated by Hatfield and Milgrom (2005). They show that substitutability (on the school side) is sufficient for the existence of a student optimal (and also school optimal) *stable* assignment. If in addition the Law of Aggregate Demand is satisfied, then the mechanism defined by the student-offering Cumulative Offer Process is *strategy-proof*. Hatfield and Kojima (2010) show the extent to which substitutability can be relaxed while maintaining existence and define two progressively weaker notions—unilateral substitutability and bilateral substitutability. Both guarantee the existence of a *stable* assignment, but for the latter, optimality for any side of agents as well as a doctor-*stategy-proof* mechanism is lost. In the context of many-to-many matching with contracts, when both sides can sign multiple contracts, Hatfield and Kominers (2017) show that substitutability of both sides choice

[13]See Roth and Peranson (1999) Table 1, and Kojima, Pathak, and Roth (2013) Table 1.

functions guarantees the existence of a *stable* assignment.

The emerging message seemed to be that straying too far (or at all) from the substitutable domain results in the non-existence of a *stable* assignment, and/or *strategy-proof* mechanisms. However, the discussion turned when a series of papers emerged identifying interesting and real-life cases where substitutability was violated, but a *stable* assignment still exists.

Sönmez and Switzer (2013) study the problem of cadet branch-of-choice matching at the United States Military Academy. They show that although branches choice functions are not substitutable, the cadet optimal assignment exists and defines a *stable* and *strategy-proof* mechanism that respects cadets' priorities. Kominers and Sönmez (2016) introduce a more general environment where unilateral substitutability is violated, and an agent optimal assignment does not exist, but there is still a *strategy-proof* mechanism that selects from the *stable* correspondence. In these two papers, bilateral substitutability was still satisfied. Abizada (2016) and Abizada and Dur (2018) study college admissions problem with stipend offers. Despite the presence of complementarities causing the failure of bilateral substitutability, they show the existence of a *pairwise stable* and *strategy-proof* mechanism exists.[14] We contribute to this literature by identifying another natural environment in which (bilateral) substitutability is violated, but a *stable* and *strategy-proof* mechanism exists.

2 Model

We consider the problem of assigning students to schools. As in the standard model, there is a finite set of students, I, and a finite set of schools, S. Unlike the standard model, the students are part of a family and each school has multiple grades. Let $G = \{1, ..., n\}$ be the finite set of grades and $\gamma : I \to G$ be the grade function such that student i applies to grade $\gamma(i)$. Let $\gamma(J) = \cup_{i \in J} \gamma(i)$.

For each grade $g \in G$, let I^g denote the set of students applying for grade g, i.e., $I^g = \{i \in I : \gamma(i) = g\}$. Let $\mathcal{F}$ be a partition of students into *families*. For a student i in

[14] In *pairwise stability*, blocking coalitions are restricted to be a single student and school pair.

partition $f \in \mathcal{F}$, we will refer to f as i's family. If i and j are in the same family, we refer to them as *siblings*. If i is the only member of i's family, then we call i an only child. To avoid technical complications, we restrict families to have either one student or two, and we do not allow there to be twins (i.e., if i and j are siblings, then they are applying for different grades). This is not without loss of generality, and we explain in Appendix A complications that these assumptions avoid.[15] Each student $i \in I$ has a strict ranking P_i over the set of schools and being unassigned (denoted $\emptyset$). Let $\mathcal{P}$ be the set of strict rankings over $S \cup \{\emptyset\}$. For each $P_i \in \mathcal{P}$, let R_i be the weak ranking associated with P_i.[16] We require that if i and j are siblings, then $P_i = P_j$ (and therefore, it is unambiguous to refer to the preferences of a family). This assumption is based on the restrictions imposed by WCPSS. They allow siblings to be treated as individuals, but if students wish to be treated as siblings in the assignment procedure, all siblings must submit the same ranking of schools. With slight abuse of notation, we represent the preference of a family f with P_f and P_f is the same as the preference of each member of family f.

Each school $s \in S$ has a capacity vector $q_s = (q_s^g)_{g \in G}$ where q_s^g denotes school s's capacity for grade $g \in G$. In addition, each school $s \in S$ has a vector of priority rankings for each grade denoted $\succ_s = (\succ_s^g)$ where $\succ_s^g$ is a strict ranking of I^g, the students applying for grade $g \in G$.

A subset of students $J \subseteq I$ is **closed under siblings** if for each family $f \in \mathcal{F}$, either $f \subseteq J$ or $f \cap J = \emptyset$.[17] In words, if J contains a student, then it also contains that student's sibling (if any). An **assignment** μ is a function $\mu : I \to S \cup \{\emptyset\}$. We refer to the assignment of student i, the students assigned to a school s, and the students assigned to grade g at s as μ_i, μ_s, and μ_s^g, respectively. Mathematically, $\mu_s = \{i \in I : \mu_i = s\}$ and $\mu_s^g = \mu_s \cap I^g$. An assignment μ is **feasible** if for each school $s \in S$ (i) μ_s is closed under siblings and (ii) for each grade $g \in G$, $|\mu_s^g| \leq q_s^g$. In words, all siblings are assigned to the same school, possibly $\emptyset$, and no school is assigned more students in a grade than it has capacity for. We restrict our attention to feasible assignments and for expositional convenience will typically refer to

[15] Empirically, these types comprise a smaller subset of the applying students: In the 2016-2018 school years, only 2.4% of the applying students were in a family with more than two siblings, and 3.4% were in a family with twins.

[16] For each $s, s' \in S$, $s \, R_i \, s'$ if and only if $s \, P_i \, s'$ or $s = s'$.

[17] Note that f can either be a set of two siblings or a set of a single student.

them simply as assignments. Let $\mathcal{A}$ be the set of all possible feasible assignments. For each assignment $\mu \in \mathcal{A}$, and each $J \subseteq I$, school s has **available seats for J at μ** if for each $g \in \gamma(J)$, $|\{j \in J : \gamma(j) = g\}| \leq q_s^g - |\mu_s^g|$.

A problem is a tuple $(I, S, G, q, \succ, P)$, and a **mechanism** (or rule) φ recommends for each problem an assignment that is feasible (for that problem). We denote the assignment selected by mechanism φ under problem $(I, S, G, q, \succ, P)$ with $\varphi(I, S, G, q, \succ, P)$ and the assignment of student i with $\varphi_i(I, S, G, q, \succ, P)$.

We say student $i \in I$ (weakly) prefers assignment μ to assignment ν if $\mu_i P_i \nu_i$ ($\mu_i R_i \nu_i$). Assignment μ **Pareto dominates** assignment ν if each $i \in I$ weakly prefers μ to ν and some $j \in I$ prefers μ to ν. Recall that we restrict two siblings to report the same preference; thus a manipulation by a family of two is necessarily comprised of changing both students' preferences. The next property states that no single student or pair of siblings is better off from reporting false preferences. A mechanism is **strategy-proof** if for each problem $(I, S, G, q, \succ, P)$, each $f \in \mathcal{F}$, each $P'_f = (P'_i)_{i \in f} \in \mathcal{P}^f$, and each $i \in f$, $\varphi_i(I, S, G, q, \succ, P) \, R_i \, \varphi_i(I, S, G, q, \succ, P'_f, P_{-f})$. If f is comprised of one student, then the definition is standard. Note that since siblings have the same preference, and are never separated, there is no need to define preferences over arbitrary pairs of assignments.

2.1 A New Criterion for Respecting Priorities

In this section, we define "justified envy" in terms of which coalitions are able to block an assignment. Here, we interpret a blocking pair as an objection by a parent (or parents) to an assignment that the school board would concur with. Without siblings and grades, student i and school s would form a blocking pair to an assignment if i prefers s to her assignment and i has a higher priority at s than one of the students j assigned to s. We can define an analogous concept in our model. To be consistent with the literature (and to differentiate from how we define *stability*) we say student $i \in I^g$ (i.e. a g^{th} grader) has **justified envy** at assignment μ if there exists a school s and a student $j \in \mu_s^g$ such that $s \, P_i \, \mu_i$ and $i \succ_s^g j$. The presence of siblings means that justified envy is not sufficient to constitute a blocking pair in our more general problem. Consider the following example.

Example 1 *Students i_1 and i_2 are siblings, while j and k are only children. School s has two grades, priorities for students are listed below each grade, and each grade has one available seat.*

$\succ_s^1$	$\succ_s^2$
i_1	k
j	i_2

Consider the assignment where j and k are assigned to s. Student i_1 has justified envy of j. However, it is not feasible to assign i_1 to s unless we also assign her sibling i_2 to s. Note that i_2 does not have sufficiently high priority at s to warrant admission over k—so assigning i_1 and i_2 to s generates justified envy from k.

It is not sufficient for one sibling to be admitted to the school as this would result in an infeasible assignment. As a result, it is not enough for one sibling to have justified envy. All siblings must have justified envy.

The presence of siblings also restricts the ability of only children to block an assignment. Consider the following example.

Example 2 *Students i_1 and i_2 are siblings, while j is an only child. School s has two grades, priorities for students are listed under each grade, and each grade has one available seat.*

$\succ_s^1$	$\succ_s^2$
i_1	j
	i_2

There are three feasible assignments:

	Grade	
	1	2
Assignment 1	i_1	i_2
Assignment 2	$\emptyset$	j
Assignment 3	$\emptyset$	$\emptyset$

The only feasible assignment in which all seats are filled is the first assignment: i_1 and i_2 are assigned to s.

In Example 2, if we assign i_1 and i_2 to school s, then student j has justified envy. However, since the student she envies has a sibling, honoring j's objection would result in more than one student being removed from the school. Since school districts care about capacity utilization far more than honoring preferences, this is not an objection they would grant. Therefore, we do not allow a set of students of size n to block the assignments of more than n students.

These two observations motivate our definition of blocking coalitions. It intuitively extends justified envy in two ways: a block must consider all siblings, and students one by one have justified envy. We also allow for students to block empty seats.

Definition 1 *A set of students $J = \{j_1, \dots, j_n\}$* ***block*** *an assignment μ if there exists a set of students $K = \{k_1, \dots, k_m\}$ and school s such that:*

1. *Both J and K are closed under siblings and $|J| \geq |K|$.*
2. *For each $x \in \{1, \dots, m\}$, j_x has justified envy of k_x at s.*
3. *s has available seats for $j_{m+1}, \dots, j_n$ at μ.*

We allow K to be empty. An assignment is ***suitable*** *if it is not blocked by any set of students. A* suitable *assignment μ is called* ***student optimal suitable*** *if there does not exist another* suitable *assignment that Pareto dominates μ.*

In Definition 1, the first condition addresses our concerns from Examples 1 and 2. Requiring J and K to be closed under siblings and $|J| \geq |K|$ means that 1) for each student $k \in K$, there is a different student in J that has justified envy over k, 2) we do not remove more students than we are assigning, and 3) honoring priorities of students in J results in a feasible assignment. Note that this definition is a generalization of justified envy if there are no siblings, conditions 1 and 3 hold trivially for any instance of justified envy. Moreover, an only child with justified envy of another only child is still sufficient to block an assignment.

The following example illustrates several new types of coalitions admitted under this.

Example 3 *We provide three new types of blocking scenarios.*

$\succ_s^1$	$\succ_s^2$
i_1	i_2
j_1	j_2

$\succ_s^1$	$\succ_s^2$
m	n
i_1	i_2

$\succ_s^1$	$\succ_s^2$	$\succ_s^3$	$\succ_s^4$	$\succ_s^5$
ℓ	i_1	i_2	k_1	k_2
m_1	m_2	j_1	j_2	n

In the left priority profile, students i_1 and i_2 would be able to block the assignment of j_1 and j_2 to s. Similarly, in the middle profile, m and n would be able to block the assignment of i_1 and i_2 to s. Our notion also allows for interesting combinations of individual and sibling pair students to form a blocking coalition. In the right profile, the combination of single and sibling pair students in the first row would be able to block the assignment of single and sibling pair students in the second row to s.

If there exists an assignment μ that has no justified envy, then μ is also *suitable*. No individual agent has justified envy, so it is not possible for any coalition to have one by one justified envy. A *suitable* assignment, however, may have an agent with justified envy. In Example 2, Assignment 1 is *suitable* but j has justified envy.

Despite our property being an intuitive generalization, several interesting properties of the set of assignments that have no justified envy do not carry over when we consider *suitable* assignments.

In the standard school choice problem, there is a unique assignment that has no justified envy and Pareto-dominates any other assignment that has no justified envy. This is no longer true of *suitable* assignments. We show by example that there is no unique student optimal *suitable* assignment.

Example 4 *Students i_1 and i_2 are siblings, as are students j_1 and j_2. School s has two grades, priorities for students are listed under each grade, and each grade has one available seat.*

$\succ_s^1$	$\succ_s^2$
i_1	j_2
j_1	i_2

There are two suitable *assignments: assigning either i_1 and i_2 to s or j_1 and j_2 to s. There is no Pareto ranking of these two assignments.*

When schools have responsive preferences, the set of unassigned students at each *stable* assignment is the same.[18] Although we have not defined *stability* yet, it is equivalent to having no justified envy when schools have such preferences. In contrast, in the example above, we observe two *suitable* assignments where the set of unassigned students is different.

Lastly, any mechanism that selects an assignment without justified envy, satisfies an "unassigned invariance" property: adding a new student to the problem does not cause some student who was unassigned (in any assignment with no justified envy) to now be assigned (in any assignment without justified envy). A mechanism that selects from the *suitable* correspondence does not always satisfy this property.

2.2 Choice Among Sets of Students

The primary challenge in our problem is that although we know how a school ranks students within a grade, we do not know how the school would choose among applicants across grades. In other words, schools provide individual student rankings—not sets of students. In practice, it is essentially left to the designer to extend priorities to selections over sets of students. Here we consider what choice rules are consistent with priorities, capacities, and the institutional constraint that siblings are assigned to the same school. Formally, let $\mathscr{P}(I)$ denote the powerset of I. A **choice rule** for school s is a function $C_s : \mathscr{P}(I) \to \mathscr{P}(I)$ such that for every $J \subseteq I$, $C_s(J) \subseteq J$. For each subset of students $J \subseteq I$, a subset $K \subseteq J$ is **closed under siblings in J** if for each $f \in \mathcal{F}$, either $f \cap J \subseteq K$ or $(f \cap J) \cap K = \emptyset$. In words, if K contains a student, then it also contains that student's sibling if they appear in J. In our context, there are two additional requirements for a choice rule to be valid: for any subset of students $J \subseteq I$ (i) for every grade g, $|I^g \cap C_s(J)| \leq q_s^g$ and (ii) $C_s(J)$ is closed under siblings in J.[19] In words, the first condition says that s does not choose more students for grade g than the grade has capacity. The second condition says that a school must choose either all siblings or none in J.

In the literature, a choice rule is defined to be **responsive** to a priority order and

[18] This is referred to as the "Rural Hospital Theorem" of Roth (1986). Also see Kojima (2011), Klijn and Yazici (2014), and Martínez, Massó, Neme, and Oviedo (2000).

[19] In general, for any $J \subseteq I$, K is a **valid set in J for school s** if $K \subseteq J$, $|I^g \cap K| \leq q_s^g$, and K is closed under siblings in J.

capacity if it chooses the highest ranked students up to the capacity. Responsive choice rules are not typically consistent with the requirement that siblings must be assigned to the same school. This is illustrated by Example 1. Given the set of students $\{i_1, j, k, i_2\}$, if school s chooses the highest ranked students for each grade, then s would choose i_1 and k. However, this chosen set is not valid as it separates siblings.

Responsiveness captures the notion that even when there is some ambiguity regarding preferences, some comparisons are unambiguous. If we want to choose two out of four students, then it is ambiguous whether having the top and last ranked student is better or worse than having the second and third ranked students; however, it is unambiguous that having the first and second ranked students is the best possible outcome. Here, we highlight a second comparison that is unambiguous. It is clear that having your first and third favorite student is better than having your second and fourth favorite student. The assignment has improved in each position.

Definition 2 *Given two disjoint sets of students $J, K \subseteq I$, and a priority order $\succ_s = (\succ_s^g)_{g\in G}$,* ***J rank-dominates K at school s*** *if there exists an ordering of $J = \{j_1, \ldots, j_n\}$ and an ordering of $K = \{k_1, \ldots, k_m\}$ such that $n \geq m$ and for every $x \leq m$ $j_x \succeq_s^{\gamma(j_x)} k_x$ and for some $x' \leq m$, $j_{x'} \succ_s^{\gamma(j_{x'})} k_{x'}$.*

A school should never choose a rank-dominated set of students—motivating the following definition.

Definition 3 *Given a set of students $I' \subseteq I$ and a priority ordering $\succ_s = (\succ_s^g)_{g\in G}$, a choice function C_s for school s* ***conforms to*** $\boldsymbol{\succ_s}$ *if there does not exist a set of students $J \subset I'$ such that (i) J is a valid set in I' for s and (ii) J rank-dominates $C_s(I')$ at s.*

Without siblings, a choice function conforming to a priority is equivalent to a choice function responding to a priority. In particular, choosing the q_s^g-highest students for each grade $g \in G$ is feasible when there are no siblings and rank-dominates any other set; therefore, it must be chosen by a choice rule that conforms to the ranking. This is also chosen by a responsive choice rule, and so the two definitions are equivalent. In the rest of our analysis, we focus on the choice functions which conform to the each school's respective priority ordering.

A choice rule is **substitutable** if for each $I'' \subset I' \subseteq I$, $i \notin C_s(I'')$ implies $i \notin C_s(I')$.

We show that a conforming choice function cannot be substitutable.

Theorem 1 *Let $s \in S$, and C_s conform to $\succ_s$. Then, C_s violates substitutability.*

Proof. Let $G = \{1, 2\}$, $S = \{s\}$ and $I = \{i_1, i_2, j, k, l, m\}$ where i_1 and i_2 are siblings. Students $\{i_2, j, k\}$ are second graders, and $\{i_1, l, m\}$ are first graders. School s has grade-capacities $q_s^1 = 1$ and $q_s^2 = 2$. Consider the following priorities for s:

$\succ_s^1$	$\succ_s^2$
m	j
i_1	k
l	i_2

We describe several situations where it is unambiguous what must be selected by any conforming choice function (as it is the only undominated alternative).

We first consider the students in $I_1 = \{j, i_1, i_2, \ell\}$. The set of students $\{i_1, i_2, j\}$ rank-dominates any valid set in I_1 for s. Hence, any conforming choice function C_s selects $\{i_1, i_2, j\}$ when I_1 is considered, i.e. $C_s(I_1) = \{i_1, i_2, j\}$.

Second, we consider the students in $I_2 = I_1 \cup \{k\}$. There are two valid sets in I_2 for s that are not rank-dominated by any other valid sets in I_2: $\{j, k, \ell\}$ and $\{i_1, i_2, j\}$. If $C_s(I_2) = \{j, k, \ell\}$, then since $\ell \in I_1 \subseteq I_2$, $\ell \notin C_s(I_1)$, and $\ell \in C_s(I_2)$, C_s is not substitutable. Then, any conforming and substitutable choice function C_s selects $\{i_1, i_2, j\}$ when I_2 is considered, i.e. $C_s(I_2) = \{i_1, i_2, j\}$.

Lastly, we consider all students $I = I_2 \cup \{m\}$. The set of students $\{j, k, m\}$ rank-dominates any valid set in I for s. Hence, any conforming choice function C_s selects $\{j, k, m\}$ when I is considered, i.e. $C_s(I) = \{j, k, m\}$. Since $k \in I_2 \subseteq I$, $k \notin C_s(I_2)$, and $k \in C_s(I)$, C_s is not substitutable. ■

The violations of substitutability are different than what we might initially have expected. A school must either accept both siblings or accept neither, so the most obvious type of complements are this type of "left shoe/right shoe" complements. Although this type of complement is clear, this is not what creates a violation in Theorem 1. These complements are "hidden" in the sense of Hatfield and Kominers (2016): Since each school either accepts both students or rejects both, and since siblings submit the same ranking, a school will never receive the application of just one sibling. Rather, what is present here is what we refer to as "the enemy of my enemy is my friend" complements. Suppose a student i, who has no siblings, is rejected in favor of j who has an older sibling. If the school receives additional applications for the higher grade, this can cause j's older sibling to be rejected which causes j to be rejected. This sequence of rejections can result in i being accepted as there is now a "free" seat at the lower grade. In this sense, even an only child can be complements with other only children as their application can help only children at other grades be accepted.

3 Existence Via A New Mechanism

Our main result demonstrates that a *suitable* assignment always exists. We introduce a new mechanism (by means of an algorithm) that selects a *suitable* assignment and is also *strategy-proof*.

The algorithm assigns students grade-by-grade, except that siblings of the students in the grade under consideration may also be assigned. Let $\triangleright$ be a strict precedence order over G; for each $g, g' \in G$, $g \triangleright g'$ means assignment to g is done before g'. Label grades so that $g_1 \triangleright g_2 \triangleright ... \triangleright g_{|G|}$. Let $I_1 = I$.

For any problem $(I, S, G, q, \succ, P)$ and any precedence order $\triangleright$, the **Sequential Deferred Acceptance w.r.t.** $\triangleright$ (***SDA***$^{\triangleright}$) selects its outcome through the following procedure:

Step 1: Deferred Acceptance with Types for Grade g_1

Step 1.0: (Type Determination for Grade g_1**)** If student $i \in I_1 \cap I^{g_1}$ has a sibling in I^g, then she is a type g student. If student i does not have any sibling, then she is type g_1 student.

Step 1.1: Each student $i \in I^{g_1}$ applies to her best choice under P_i. Each school s first considers each type $g \in G$ applicant and tentatively accepts the best q_s^g of the type g

applicants according to $\succ_s^{g_1}$ and rejects the rest type g applicants. Among the unrejected applicants, each school s tentatively accepts the best $q_s^{g_1}$ applicants according to $\succ_s^{g_1}$ and rejects the rest.

For each m>1:

Step 1.m: Each student $i \in I^{g_1}$ applies to her best choice under P_i that has not rejected her yet. Each school s first considers each type $g \in G$ applicant and tentatively accepts the best q_s^g type g applicants according to $\succ_s^{g_1}$ and rejects the rest of the type g applicants. Among the unrejected applicants, each school s tentatively accepts the best $q_s^{g_1}$ applicants according to $\succ_s^{g_1}$ and rejects the rest.

Step 1 terminates when there is no more rejection. Each student i and her sibling, if any, are assigned to the school, possibly $\emptyset$, tentatively holding i when Step 1 terminates. Each assigned student is removed and we denote the remaining students with I_2. We update the number of remaining seats in each school and grade.

Step k>1: Deferred Acceptance with Types for Grade g_k

Step k.0: (Type Determination for Grade g_k**)** If student $i \in I_k \cap I^{g_k}$ has a sibling in I^g, then she is a type g student. If student i does not have any siblings, then she is type g_k student.

Step k.1: Each student $i \in I^{g_k}$ applies to her best choice under P_i. Each school s first considers each type $g \in G$ applicant and tentatively accepts the best q_s^g type g applicants according to $\succ_s^{g_k}$ and rejects the rest of the type g applicants. Among the unrejected applicants, each school s tentatively accepts the best $q_s^{g_k}$ applicants according to $\succ_s^{g_k}$ and rejects the rest.

For each m>1:

Step k.m: Each student $i \in I^{g_k}$ applies to her best choice under P_i that has not rejected her yet. Each school s first considers each type $g \in G$ applicant and tentatively accepts the best q_s^g type g applicants according to $\succ_s^{g_k}$ and rejects the rest of the type g applicants. Among the unrejected applicants, each school s tentatively accepts the best $q_s^{g_k}$ applicants according to $\succ_s^{g_k}$ and rejects the rest.

Step k terminates when there is no more rejection. Each student i and her sibling, if any, are assigned to the school, possibly $\emptyset$, tentatively holding i when Step k terminates. Each assigned student is removed and we denote the remaining students with I_{k+1}. We update the number of remaining seats in each school and grade.

The algorithm terminates after we run DA for all grades, i.e. after Step $|G|$.

Notice that each order $\triangleright$ of the grades defines a different mechanism. Next, we illustrate the dynamics of $SDA^{\triangleright}$ in the following example.

Example 5 *Let* $I = \{i_1, i_2, j_1, j_2, k_1, k_2, \ell_1, \ell_2, m, n, o\}$, $S = \{s_1, s_2, s_3\}$, $G = \{1, 2, 3\}$ *and* $1 \triangleright 2 \triangleright 3$. *For each* $x \in \{i, j, k, \ell\}$, *students* x_1 *and* x_2 *are siblings. Let* $I^1 = \{i_1, j_1, k_1, m\}$, $I^2 = \{i_2, j_2, \ell_1, n\}$, $I^3 = \{k_2, \ell_2, o\}$, $q_{s_1} = (2, 1, 1)$, $q_{s_2} = (1, 2, 1)$ *and* $q_{s_3} = (1, 1, 1)$. *Let preferences and priorities be as below:*

P_i	P_j	P_k	P_ℓ	P_m	P_n	P_o
s_1	s_1	s_2	s_3	s_1	s_1	s_3
s_2	s_2	s_3	s_2	s_3	s_3	s_2
s_3	s_3	s_1	s_3	s_2	s_2	s_1

	s_1			s_2			s_3	
$\succ^1_{s_1}$	$\succ^2_{s_1}$	$\succ^3_{s_1}$	$\succ^1_{s_2}$	$\succ^2_{s_2}$	$\succ^3_{s_2}$	$\succ^1_{s_3}$	$\succ^2_{s_3}$	$\succ^3_{s_3}$
i_1	i_2	k_2	j_1	ℓ_1	k_2	k_1	n	k_2
j_1	j_2	o	i_1	i_2	o	m_1	ℓ_1	o
m	ℓ_1	ℓ_2	k_1	j_2	ℓ_2	j_1	j_2	ℓ_2
k_1	n		m	n		i_1	i_2	

$SDA^{\triangleright}$ finds its outcome as follows:

Step 1.0: *Students i_1 and j_1 are type 2, student k_1 is type 3, and student m is type 1.*

Step 1.1: *Students i_1, j_1 and m apply to s_1 and student k_1 applies to s_2. School s_1 first considers type 2 applicants (i_1 and j_1) and rejects j_1 since $q^2_{s_1} = 1$ and $i_1 \succ^1_{s_1} j_1$. Then, two remaining applicants (i_1 and m) are tentatively accepted by s_1. Since k_1 is the only applicant for s_2 and its quota is not binding for grade 1 and grade 3, s_2 tentatively accepts k_1.*

Step 1.2: *Students i_1 and m apply to s_1, and students k_1 and j_1 apply to s_2. Only student k_1 is rejected from s_2, and all the other students are tentatively accepted.*

Step 1.3: *Students i_1 and m apply to s_1, student j_1 applies to s_2, and student k_1 applies to s_3. Step 1 terminates since no student is rejected. Students i_1, i_2, and m are assigned* *to s_1, students j_1 and j_2 are assigned to s_2, and students k_1 and k_2 are assigned to s_3. The updated quotas are: $q_{s_1} = (0, 0, 1)$, $q_{s_2} = (0, 1, 1)$ and $q_{s_3} = (0, 1, 0)$.*

Step 2.0: *Student ℓ_1 is type* 3 *and student n is type 2. All other students in I^2 were assigned in Step 1.*

Step 2.1: *Students ℓ_1 and n apply to s_3 and s_1, respectively. Student ℓ_1 is rejected because there is no remaining seat for her sibling for grade 3 and student m is rejected since all seats of s_1 were allocated in Step 1.*

Step 2.2: *Students ℓ_1 and n apply to s_2 and s_3, respectively. Step 2 terminates since no student is rejected. Student ℓ_1 and ℓ_2 are assigned to s_2 and student m is assigned to s_3. The updated quotas are: $q_{s_1} = (0, 0, 1)$, $q_{s_2} = (0, 0, 0)$ and $q_{s_3} = (0, 0, 0)$.*

Step 3.0: *Student o is type* 3. *All other students in I^3 were assigned in Steps 1 and 2.*

Step 3.1: *Student o applies to s_3 and she is tentatively accepted. Step 3 terminates since no student is rejected. Student o is assigned to s_3.*

Now we are ready to present the properties of $SDA^{\triangleright}$.

Theorem 2 *For each order $\triangleright$, the $SDA^{\triangleright}$ is* suitable *and* strategy-proof.

Proof. Consider an arbitrary problem $(I, S, G, q, \succ, P)$ and an order over G denoted by $\triangleright$ where $g_\ell \triangleright g_{\ell+1}$ for each $\ell \in \{1, ..., |G| - 1\}$. Let $\mu = SDA^{\triangleright}(I, S, G, q, \succ, P)$.

Suitability: We prove by induction and show that μ cannot be blocked by some valid set of students. We first consider the set of students who are assigned in Step 1 of the $SDA^{\triangleright}$ algorithm, i.e. the students in I^{g_1} and their siblings from other grades. If there exist students $i, j \in I^{g_1}$ such that $\mu_j P_i \mu_i$ and $i \succ_s^{g_1} j$, then i has a sibling applying for some grade $g' \neq g_1$ and all available seats of μ_j at grade g' are filled with students whose siblings have higher priority than i under $\succ_{\mu_j}^{g_1}$, and they are also assigned to s. Therefore, μ cannot be blocked by a group of students that includes the students assigned in Step 1. Note that the set of students assigned in Step 1 includes all students in I^{g_1} and their siblings.

Suppose μ cannot be blocked by a set of students that includes the students assigned in the first k steps of $SDA^{\triangleright}$. We consider the students assigned in Step $k+1$. By our inductive hypothesis and Definition 1, it suffices to consider the subproblem with the updated quotas that we have in Step $k+1$ of the SDA algorithm. If there exist students $i, j \in I^{g_{k+1}}$ such that $\mu_j P_i \mu_i$ and $i \succ_s^{g_{k+1}} j$, then either i has a sibling applying for some grade $g' \neq g_{k+1}$ and all available seats of μ_j at grade g' are filled with students whose siblings have higher priority than i under $\succ_{\mu(j)}^{g_{k+1}}$ and they are also assigned to s or j has been assigned in some step $k' \leq k$ and j has a sibling in $I^{k'}$. Therefore, μ cannot be blocked by a group of students $\bar{I}$ which includes the students assigned in Step $k+1$.

Since the algorithm for $SDA^{\triangleright}$ ends after a finite number of steps ($|G|$), we are done.

Strategy-proofness: Consider Step 1 of the algorithm for $SDA^{\triangleright}$. We will show that students in $I_1 \cap I^{g_1}$ cannot manipulate. Let $t : I_1 \cap I^{g_1} \to G$ identify for each agent a type as follows: For each $i \in I_1 \cap I^{g_1}$, let $t(i)$ be g_1 if i has no siblings, and g_k if i has a sibling in grade $g_k \in G$. For each $s \in S$, and each $g \in G$, let $\hat{q}_s^g = \min\{q_s^{g_1}, q_s^g\}$. Then, the tuple $(I_1 \cap I^{g_1}, S, (q_s^{g_1})_{s\in S}, (\succ_s^{g_1})_{s\in S}, P_{I_1\cap I^{g_1}}, t, (\hat{q}_s^g)_{g\in G, s\in S})$ forms a school choice problem with type-specific quotas as in Abdulkadiroğlu and Sönmez (2003) (what they refer to as controlled choice with flexible constraints). The assignment at the end of Step 1 of the $SDA^{\triangleright}$ is then the same as the outcome of their modification of the Deferred Acceptance mechanism. Notice that no agent in $I_1 \cap I^{g_1}$ is involved in the algorithm again. By Proposition 5 of Abdulkadiroğlu and Sönmez (2003), no agent in $I_1 \cap I^{g_1}$ can manipulate their mechanism, and the same follows for the $SDA^{\triangleright}$. Let I_2 be the remaining students after Step 1 is implied.

We repeat this procedure for Step 2; the remaining steps are similar. Let $t : I_2 \cap I^{g_2} \to G$ identify for each agent a type as follows: For each $i \in I_2 \cap I^{g_2}$, let $t(i)$ be g_2 if i has no siblings, and g_k if i has a sibling in grade $g_k \in G\backslash\{g_1\}$. For each grade $g \in G\backslash\{g_1\}$, let X_s^g be the number of students assigned to grade g at school s in Step 1 of the $SDA^{\triangleright}$. For each $s \in S$, and each $g \in G\backslash\{g_1\}$, let $\hat{q}_s^g = \min\{q_s^{g_2} - X_s^{g_2}, q_s^g - X_s^g\}$. Then, the tuple $(I_2 \cap I^{g_2}, S, (q_s^{g_2} - X_s^{g_2})_{s\in S}, (\succ_s^{g_2})_{s\in S}, P_{I_2\cap I^{g_2}}, t, (\hat{q}_s^g)_{g\in G, s\in S})$ forms a school choice problem with type-specific quotas. By repeating the same reasoning for all grades, we can show that no student can manipulate the $SDA^{\triangleright}$. ■

Although $SDA^{\triangleright}$ is *suitable* and *strategy-proof*, it is not *Pareto-efficient*. This is not so suprising, as in the standard school choice problem, the Deferred Acceptance mechanism is not *Pareto-efficient* either.[20]

We may ask whether there exists a *strategy-proof* and *suitable* mechanism whose assignment is not Pareto-dominated by any other *suitable* assignment in any problem. Unfortunately, our next proposition demonstrates that no such mechanism exists.

Proposition 1 *There is no* strategy-proof, suitable, *and Pareto-undominated (within the* suitable *set) mechanism.*

Proof. Suppose by contradiction that there is such a mechanism; call it φ. Let $I = \{i_1, i_2, j_1, j_2, k_1, k_2, \ell_1, \ell_2\}$, $S = \{s_1, s_2, s_3\}$, and $G = \{1, 2\}$. At each grade, each school has a capacity of one. Let preferences and priorities be as below:

P_i	P_j	P_k	P_ℓ
$\boldsymbol{s_2}$	$\boldsymbol{s_1}$	s_1	$\boldsymbol{s_3}$
s_1	s_2	$\boldsymbol{\emptyset}$	s_1

s_1		s_2		s_3	
$\succ^1_{s_1}$	$\succ^2_{s_1}$	$\succ^1_{s_2}$	$\succ^2_{s_2}$	$\succ^1_{s_3}$	$\succ^2_{s_3}$
ℓ_1	ℓ_2	j_1	j_2	k_1	k_2
i_1	i_2	i_1	i_2	ℓ_1	ℓ_2
k_1	j_2				
j_1	k_2				

The only *suitable* assignment that is not Pareto-dominated by some other *suitable* assignment is bolded.

Consider a second problem that is the same except that the j sibling pair reports $P'_{j_1} = P'_{j_2}$ where $s_1 \, P'_{j_1} \, \emptyset \, P'_{j_1} \, s_2 \, P'_{j_1} \, s_3$. By *strategy-proofness*, the same assignment is selected by φ.

Finally, consider a third problem that is the same as the second except that the k sibling pair reports $P'_{k_1} = P'_{k_2}$ where $s_1 \, P'_{k_1} \, s_3 \, P'_{k_1} \emptyset$. By *strategy-proofness*, φ does not assign the k siblings to s_1. By *suitability*, φ assigns s_3 to k_1 and k_2. By feasibility and *suitability*, φ assigns s_1 to ℓ_1 and ℓ_2, and s_2 to i_1 and i_2. From this assignment, notice that k and ℓ are better off swapping schools. If they do so, this forms a *suitable* and Pareto-dominating assignment—contradicting the assumptions on φ. ■

[20] Also, in the standard school choice problem there is no mechanism that has no justified envy and is *Pareto-efficient*.

4 Comparison of the Naive and Sequential Deferred Acceptance Assignments In Wake County

Using data from the WCPSS magnet program assignment for the 2018-2019 school year, we present a comparison of the assignment outcomes of the grade-by-grade Deferred Acceptance (the "naive" assignment) and the Sequential Deferred Acceptance mechanisms in Table 2. A total of 6,994 students applied for seats across grades K-12.

For each level of schooling (elementary, middle, and high school), under the $SDA^{\triangleright}$ mechanism we start with the highest grade and continue with the next highest grade. We augment the $SDA^{\triangleright}$ to accommodate for twins and family group sizes of more than two.[21] If there were no such students, our formal definition never separates siblings; but here, we observe several mismatches. We use this augmentation for two reasons: it is simple and transparent, and the resulting mismatch may be seen as a conservative estimate regarding the efficacy of the $SDA^{\triangleright}$ (as we could make further changes ensuring that twins always go to the same school).

The key finding is that the naive assignment separates siblings at a rate 15 times that of the $SDA^{\triangleright}$. Furthermore, our new assignment also demonstrates that we can implement the institutional constraint of keeping siblings together with very few students causing priority violations—only 17 such instances occurred.

The naive assignment does not meet the WCPSS policy of sending all siblings to the same school. This is perhaps unsurprising as the mechanism does not take into account the existence of siblings at all. In total over all grades K-12, 171 students are separated from their siblings, or 12.6% of the total number of applying students with siblings (1,361). This percentage is similar for all sub-levels of schooling: elementary (12.5%), middle (9.2%), and high (15.9%). Much of the mismatch comes from students with siblings in entry grades.

By design, the $SDA^{\triangleright}$ sends virtually all siblings to the same school. In total over all grades K-12, 99% of all siblings are assigned to the same school as their siblings; this

[21] In 2018, the average percentage of students with one sibling that is not a twin, students with one sibling that is a twin, and students with two or more siblings was respectively 16.9%, 4%, and 2.5%. For twins we treat each as a single student in the algorithm. Any separation of twins also recorded as sibling mismatch. For each student with more than one sibling, when we process the grade in which the oldest sibling appears, we also assign their younger siblings to the same school.

amounts to separating siblings 15 times less than the naive assignment. All mismatched siblings under $SDA^{\triangleright}$ are twins.

The total number of siblings who are all assigned to the same school (as opposed to all being unassigned) is also greater in the $SDA^{\triangleright}$. Over all grades K-12, it is 13% more (573 to 662; of the total number of siblings) than in the naive assignment. This captures the scenario where an older sibling is assigned to a school and thus their younger sibling now follows (but would have been unassigned in the naive assignment).

We now examine the extent to which the constraint of keeping siblings together causes instances of individual justified envy. We count the number students j for which there is another student i who has justified envy of j at the $SDA^{\triangleright}$ assignment. For the entire school district, there were only 17 such instances. Each violation of priorities is caused by the situation that we would expect: a younger sibling follows an older sibling, but individually has a lower priority than some other students who would rather attend. Of all instances, 6 are caused by students who are sibling triples, and the remaining are caused by students who are part of a sibling pair.

5 Matching with Contracts

The matching with contracts framework of Hatfield and Milgrom (2005) is a natural alternative environment to study school assignment with siblings. We may think of each seat at a school as an agreement or "contract" signed between a parent and the school for the upcoming year. Parents want seat(s) for their children, and schools select multiple students; each side has preferences described by choice functions over contracts. If a school selects a certain subgroup of students out of a set of applicants, then this may be interpreted as the selected students having the "right" to attend the school over those rejected. Thus, if an assignment is not *stable*, then there may exist a group of parents with legal recourse to claim different seats than those recommended to them.

In the matching with contracts literature, substitutability is a critical property for existence of *stable* assignments. Hatfield and Kominers (2017) show that the domain of substitutable choice functions is a maximal one guaranteeing the existence of *stable* assignments in many-to-many matching with contracts framework.

Table 2: WCPSS 2018-2019 Comparison of Naive and SDA Assignments

Grade	**Total Applicants**	**Applicants with Siblings**	**Assigned w/ Sibling**		**Mismatched Siblings**		**Justified Envy**	
			Naive	SDA	Naive	SDA	Naive	SDA
K	1,342	374	247	250	42	0	0	0
1	308	107	4	7	7	0	0	2
2	260	88	5	7	13	0	0	1
3	269	100	5	6	24	0	0	1
4	234	90	4	4	13	0	0	0
5	154	65	1	1	4	0	0	0
$\boldsymbol{K-5}$	**2,567**	**824**	**266**	**275**	**103**	**0**	**0**	**4**
6	1,785	204	156	163	17	7	0	7
7	164	28	10	11	1	0	0	0
8	121	29	16	19	6	0	0	0
$\mathbf{6-8}$	**2,070**	**261**	**182**	**193**	**24**	**7**	**0**	7
9	1,871	215	159	170	22	4	0	2
10	336	36	6	12	15	0	0	4
11	115	15	4	5	4	0	0	0
12	35	10	4	7	3	0	0	0
$\mathbf{9-12}$	**2,357**	**276**	**173**	**194**	**44**	**4**	**0**	**6**
$\boldsymbol{K-12}$	**6,994**	**1,361**	**573**	**662**	**171**	**11**	**0**	**17**

The purpose of this section is to demonstrate the existence of a real-life mechanism design application where the choice functions of *both* sides violate substitutability, and yet *stable* assignments still exist.

5.1 Existence Despite Complementarities

Let $(I, S, G, q, \succ, P)$ be a problem as before with minor differences. In particular, under this model I is the set of parents and P is the preference of parents over schools. We allow a school to prioritize parents differently for different grades, i.e. $i \succ_s^g j \succ_s^{g'} i$. We denote the subset of grades parent i is applying to with $\gamma(i)$. Consistent with Section 2, we assume $|\gamma(i)| \leq 2$ for all $i \in I$. Let $X \subseteq I \times S \times G$ be the finite set of possible contracts such that

$X = \cup_{i\in I}\left(\cup_{s\in S}\left(\cup_{g\in\gamma(i)}(i,s,g)\right)\right)$. Each contract $x \in X$ specifies a parent i, a school s, and a grade g. Let $\mathrm{i}(x)$, $\mathrm{s}(x)$, and $\mathrm{g}(x)$ denote the parent, school, and grade related to contract x, respectively. Similarly, for any subset of contracts $Y \subseteq X$, $\mathrm{i}(Y)$, $\mathrm{s}(Y)$, and $\mathrm{g}(Y)$ denote the set of parents, set of schools and set of grades related to contracts in Y, respectively. Given a subset of contracts $Y \subseteq X$, let Y_i, Y_s, and Y^g denote the contracts related to parent i, school s, and grade g, respectively.

An **assignment** $Y \subset X$ is a set of contracts such that each parent $i \in I$ appears in at most $|\gamma(i)|$ contracts and each (s,g) pair appears in at most q_s^g contracts. We refer to Y_i as i's assignment. A mechanism φ recommends for each possible problem an assignment.

For each $a \in I \cup S$, choice of agent a, denoted by $C_a : X \rightrightarrows X_a \cup \emptyset$, is a function such that for each $Y \subseteq X$, $C_a(Y) \subseteq Y_a$. Here, $C_a(Y) = \emptyset$ means that a rejects all contracts in Y.

For any set of contracts, the chosen subset for each parent is determined as follows: Parent i first considers only schools that give each of their children a seat, then, from these schools, chooses their most preferred one. Formally, given a subset of contracts $Y \subseteq X$, let $\mathrm{s}_i(Y)$ be the set of schools such that $(i,s,g) \in Y$ for all $g \in \gamma(i)$ and $s\, P_i\, \emptyset$. Each parent $i \in I$ has the following choice function:

$$C_i(Y) = \begin{cases} \{\cup_{g\in\gamma(i)}(i,s,g) \subseteq Y : s\, R_i\, s' \text{ for all } s, s' \in \mathrm{s}_i(Y)\} & \text{if } \mathrm{s}_i(Y) \neq \emptyset \\ \emptyset & \text{if } \mathrm{s}_i(Y) = \emptyset \end{cases}.$$

Let $R_i(Y) = Y \setminus C_i(Y)$ be the set of rejected contracts by parent i from Y.

We now turn to schools' choice functions. For each school s, based on its priorities and capacities, we define the **sequential choice function of** s. Given a set of contracts (possibly spanning multiple grades), the school selects a subset in exactly the same way each school tentatively accepts students in each step of the Sequential Deferred Acceptance mechanism's algorithm. That is, it proceeds iteratively grade-by-grade and narrows down the pool of contracts/students in a two-step process. First, for each subgroup of students with a sibling in a particular downstream grade, select the highest priority students; this guarantees feasibility of any selection in the second step. Second, out of the remaining students (who may have siblings across various grades), select the highest priority. We provide a formal description in Appendix B.

An assignment $Y \subset X$ is **stable** if it is

- (Individually Rational) for all $a \in I \cup S$, $C_a(Y) = Y_a$, and
- (Unblocked) There does not exist $Z \subset X$ such that $Z \neq \emptyset$, $Z \cap Y = \emptyset$, and for all $a \in \mathrm{i}(Z) \cup \mathrm{s}(Z)$, $Z_a \subseteq C_a(Y \cup Z)$.

If there is such a $Z \subset X$ satisfying the above, then we say that Z **blocks** Y (at this problem).

A mechanism that selects a *stable* assignment for each problem is *stable.*

We define two properties of choice functions that are crucial to the existence of *stable* and *strategy-proof* mechanisms in the matching with contracts literature. The first states that a contract y that is rejected from a menu of available contracts is still rejected if another contract y' is added to the menu. Intuitively, this rules out complementarities between contracts. A choice function C_a satisfies **substitutability** if for all $Y \subset X$, and $y, y' \in X \backslash Y$, $y \notin C_a(Y \cup \{y\})$ implies $y \notin C_a(Y \cup \{y, y'\})$. The second states that the number of contracts chosen (weakly) increases as the menu size grows. A choice function C_a satisfies the **Law of Aggregate Demand** (LAD) if for all $Y \subseteq Y' \subseteq X$ we have $|C_a(Y)| \leq |C_a(Y')|$.

Next, we examine whether choice functions defined above satisfy these two properties.

Proposition 2 *For each $i \in I$, C_i satisfies LAD but not substitutability.*

The proof of Proposition 2 and all other omitted proofs are given in Appendix B.

The example used in the proof of Proposition 2 can be used to show that C_i does not satisfy the weaker notion of bilateral substitutability of Hatfield and Kojima (2010).[22]

Proposition 3 *If C_s is the sequential choice function of s, then C_s satisfies neither substitutability nor LAD.*

[22] A choice function satisfies **bilateral substitutability** if for each $z, z' \in X$ and $Y \subseteq X$ such that $\mathrm{i}(z), \mathrm{i}(z') \notin \mathrm{i}(Y)$, $z \notin C_a(Y \cup z) \Rightarrow z \notin C_a(Y \cup z \cup z')$.

The same example used in the proof of Proposition 3 can be used to show that C_s does not satisfy the weaker notion of bilateral substitutability.[23]

Despite the negative results in Propositions 2 and 3, we will show that there is a *stable* and *strategy-proof* mechanism when schools have sequential choice functions. The idea is essentially the same as that of the Sequential Deferred Acceptance, but we write it in the language of contracts. We run the Cumulative Offer Process iteratively one grade at a time assigning students (from this grade) and their siblings; before the start of the next grade, capacities are revised down. Each school's effective choice function restriced to each iteration/grade (defined below) is *substitutable* and satisfies LAD, leading to its *stability* and *strategy-proofness*.

For each $s \in S$, each $g \in G$, and each grade-specific capacity vector $\hat{q} = (\hat{q}^{g'})_{g' \in G}$, let a **component choice function for $\boldsymbol{s}$ w.r.t. $\boldsymbol{g}$ and $\boldsymbol{\hat{q}}$** be a mapping $D_s^{g,\hat{q}} : \mathscr{P}(I^g) \to \mathscr{P}(I^g)$ that selects applicants from $\bar{I} \subseteq I^g$ as follows:

Step 1 For each $g' \in G \backslash \{g\}$, if the number of applicants from $\bar{I} \cap I^{g'}$ is more than $\hat{q}^{g'}$, then only the best $\hat{q}^{g'}$ applicants according to $\succ_s^g$ are tentatively kept and the rest are rejected.

Step 2 Among the unrejected ones in $\bar{I}$ in Step 1, accept the best $\hat{q}^g$ applicants according to $\succ_s^g$.

Proposition 4 *For each $s \in S$, each $g \in G$, and any capacity vector $\hat{q} \in \mathbb{N}^G$, $D_s^{g,\hat{q}}$ satisfies both substitutability and LAD.*

We define the **Sequential Cumulative Offer Mechanism w.r.t. $\triangleright$, $\boldsymbol{SCO^{\triangleright}}$** as follows: Let $\triangleright$ be a precedence order over the grades, e.g. $g_1 \triangleright g_2 \triangleright \cdots \triangleright g_{|G|}$. We will determine an assignment as a sequence of many-to-one matching with contracts problems where we process grades in order $\triangleright$.

Consider the subproblem at grade 1—I^{g_1}, schools S, preference profile $P^{g_1} = (P_i)_{i \in I^{g_1}}$, capacity profile $q^{g_1} = (q_s^{g_1})_{s \in S}$, and priority profile $\succ^{g_1} = (\succ_s^{g_1})_{s \in S}$. Treating this as a many-to-one matching problem, run the Cumulative Offer Process where each school s uses the

[23] To show violation of the property by C_s, let $Y = Y''$, $z = (j, s, 2)$, and $z' = (k, s, 1)$ from the example in the proof.

choice function $D_s^{g_1,q}$. Let μ^{g_1} be the resulting outcome, that is $\mu^{g_1} : I^{g_1} \to S \cup \emptyset$ assigns each parent i to school s—the interpretation being that each i is assigned at each of the grades $\gamma(i)$. Then, the contracts including i and $\mu^{g_1}(i)$ are selected as long as $\mu^{g_1}(i) \in S$.

We revise down capacities for each other grade. For each school $s \in S$, and grade $g' \in G \backslash g_1$, let the capacity now be $\hat{q}_s^{g'} = q_s^{g'} - |\{i \in I^{g_1} : \ \mu^{g_1}(i) = s \ and \ g' \in \gamma(i)\}|$.

Consider the subproblem at grade 2—$I^{g_2} \backslash I^{g_1}$, schools S, preference profile $P^{g_2} = (P_i)_{i \in I^{g_2} \backslash I^{g_1}}$, capacity profile $\hat{q}^{g_2} = (\hat{q}_s^{g_2})_{s \in S}$, and priority profile $\succ^{g_2} = (\succ_s^{g_2})_{s \in S}$. Treating this as a many-to-one matching problem, run the Cumulative Offer Process where each school s uses the choice function $D_s^{g_2,\hat{q}}$. Let μ^{g_2} be the resulting outcome.

Repeat this procedure for the rest of the grades to arrive at $(\mu^g)_{g \in G}$. Let $\hat{\gamma}(i)$ be the $\triangleright$-earliest grade in $\gamma(i)$. Finally, let $SCO^{\triangleright}$ for this problem select

$$\bigcup_{\substack{i \in I: \\ \mu^{\hat{\gamma}(i)}(i) \neq \emptyset}} \bigcup_{g \in \gamma(i)} (i, \mu^{\hat{\gamma}(i)}(i), g).$$

Theorem 3 *Consider an arbitrary precedence $\triangleright$. If each school s has the sequential choice function $C_s^{\triangleright}$, then the $SCO^{\triangleright}$ mechanism is* stable *and* strategy-proof.

This result is surprising, since neither the parents' nor the the schools' choice functions satisfy substitutability. This is in stark contrast to Hatfield and Kominers (2017), where they show that the *substitutable* domain is a maximal domain to guarantee existence of a *stable* assignment. The first key difference is that the grade and sibling structure *forces* schools' preferences to have such complementarities; hence, each schools' sequential choice function falls entirely outside the *substitutable* domain. The second key difference is that in our environment, while priorities vary, the grade and sibling structure is uniform across schools, and thus the exact same types of resulting complementarities appear in each school's choices. This uniform structure is crucial for our result.

6 Conclusion

We demonstrate that the seemingly trivial institutional constraint of keeping siblings together actually forces a careful consideration of grades and sibling structure. From a theoretical standpoint, sibling constraints 1) render the previous requirement of no justified envy inadequate, 2) introduce interesting complementarities in terms of schools selections across grades, and 3) require the development of new solution concepts and mechanisms. There also remain open questions regarding more general sibling structures (e.g. twins, more than two sibilings).

Essentially all school districts have various types of sibling constraints. We argue that for many, adoption of the Sequential Deferred Acceptance mechanism is an appropriate, systematic, and fair alternative to heuristic measures used to "patch" the naive assignment (to meet the constraints).

Our work also contributes to the ongoing conversation regarding *stable* assignments and complementarities in the matching with contracts literature: we provide another real-world environment where *stable* assignments exist outside (bilateral) substitutable domains.

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Appendix A

We discuss two extensions to the model: twins, and allowing for more than two siblings in each family.

Suppose that we allow for twins and require that twins cannot be separated. The following example shows that there may not be a *suitable* assignment.

Example 6 *Let* $I = \{i_1, i_2, j, k, \ell\}$*,* $S = \{s_1, s_2, s_3\}$*,* $G = \{1\}$*,* $q^1_{s_1} = 2$*, and* $q^1_{s_2} = q^1_{s_3} = 1$*. Here,* i_1 *and* i_2 *are twin. Let preferences and priorities be as below:*

P_{i_1}	P_{i_2}	P_j	P_k	P_ℓ
s_1	s_1	s_2	s_1	s_3
$\emptyset$	$\emptyset$	s_1	s_2	s_1
		s_3	$\emptyset$	$\emptyset$

$\succ^1_{s_1}$	$\succ^1_{s_2}$	$\succ^1_{s_3}$
j	k	j
ℓ	j	ℓ
i_1		
i_2		
k		

Let μ *be a* suitable *assignment. If* k *is assigned to either* s_3 *or* $\emptyset$*, then* k *blocks* μ*. So in any* suitable *assignment,* k *is assigned to either* s_1 *or* s_2*.*

Case 1: *k is assigned to s_1 in μ. Then j is assigned to s_2, otherwise j blocks (at s_1). By feasibility, i_1 and i_2 cannot be separated, so they are unassigned at μ. Then ℓ is assigned to s_3, otherwise ℓ blocks at s_3. Notice that i_1 and i_2 now blocks k at s_1, contradicting the* suitability *of* μ*.*

Case 2.1: *k is assigned to s_2 in μ, and j is assigned to s_1. Then ℓ is assigned to s_3, otherwise ℓ blocks at s_3. By feasibility, i_1 and i_2 are unassigned. Notice k blocks at s_1, contradicting the* suitability *of* μ*.*

Case2.2: *k is assigned to s_2 in μ, and ℓ is assigned to s_1. By feasibility, i_1 and i_2 are unassigned. Then j is assigned to s_1, otherwise j blocks at s_1. Notice that ℓ blocks at s_3, contradicting the* suitability *of* μ*.*

Case 2.3: *k is assigned to s_2 in μ, and i_1 and i_2 are assigned to s_1. Then j is assigned to s_3, otherwise j blocks at s_3. This leaves ℓ unassigned. Notice that j and ℓ block i_1 and i_2 at s_1, contradicting the* suitability *of* μ*.*

Similarly, suppose that we allow for families of size larger than two and require that they cannot be separated. The following example shows that the natural extension of our algorithm that accounts for three siblings may not result in a *suitable* assignment.

Example 7 *Let* $I = \{i_1, i_2, j_1, j_2, j_3, k_1, k_3\}$, $S = \{s_1, s_2\}$, $G = \{1,2,3\}$, $q_{s_1} = (2,1,1)$, *and* $q_{s_2} = (1,1,1)$. *For each* $h_x \in I$, *let* $\gamma(h_x) = x$. *Let preferences and priorities be as below: Abusing notation, we express* $P_{i_1} = P_{i_2}$ *as* P_i, *and similarly so for other sibling pairs/triples.*

P_i	P_j	P_k	$\succ^1_{s_1}$	$\succ^2_{s_1}$	$\succ^3_{s_1}$	$\succ^1_{s_2}$	$\succ^2_{s_2}$	$\succ^3_{s_2}$
s_2	s_1	s_1	i_1	i_2	j_3	j_1	j_2	j_3
s_1		s_2	j_1	j_2	k_3	k_1	i_2	k_3
			k_1			i_1		

Let $\triangleright$ *be such that* $g_1 \triangleright g_2$. *We follow the steps for the* $SDA^{\triangleright}$, *and extend the algorithm naturally when the three siblings appear. We skip Step 1.0.*

Step 1.1: j_1 *and* k_1 *apply to* s_1, *and* i_1 *applies to* s_2. *Since it is not feasible to assign* $\{j_2, k_2\}$ *to grade 2 at* s_1, *school* s_1 *rejects the lowest priority student in* $\{j_1, k_1\}$ *according to* $\succ^1_{s_1}$ *which is* k_1.

Step 1.2: k_1 *applies to* s_2. *Since it is not feasible to assign* $\{i_2, k_2\}$ *to grade 2 at* s_2, *school* s_2 *rejects the lowest priority student in* $\{i_1, k_1\}$ *according to* $\succ^1_{s_2}$ *which is* i_1.

Step 1.3: i_1 *applies to* s_1. *Since it is not feasible to assign* $\{i_2, j_2\}$ *to grade 2 at* s_1, j_1 *is rejected.*

The final assignment is $\mu_{i_1} = \mu_{i_2} = s_1$, $\mu_{k_1} = \mu_{k_2} = s_2$, *and* $\mu_{j_1} = \mu_{j_2} = \mu_{j_3} = \emptyset$. *Note that we keep in the "sequential" nature of* $SDA^{\triangleright}$ *by only using* $\succ^1_{s_1}$ *and* $\succ^1_{s_2}$ *at Step 1. The only extra comparison required is at Step 1.1 and Step 1.3, which have "reasonable" rejections in the spirit of* $SDA^{\triangleright}$. *Since it is feasible for* k_1 *and* k_2 *attend* s_1, *this assignment is blocked.*

Appendix B

For each school s, we define the **sequential choice function of s**. Each school s considers grades sequentially according to a precedence order $\triangleright$ where $g \triangleright g'$ means grade g will be processed before grade g'. We denote this choice function with $C_s^{\rhd}$ and for any given set of contracts Y, the chosen set is calculated as follows:

Step 0: Let $Y^1 = Y_s$. Let $g_k \rhd g_{k+1}$ for all $k \in \{1, 2, ..., |G| - 1\}$. Let $R_s^{\rhd}(Y) = C_s^{\rhd}(Y) = \emptyset$. Let $\bar{q}_s^{g_k} = q_s^{g_k}$ for all $k \in \{1, 2, ..., |G|\}$.

Step 1: Grade g_1 Selection

(Type Determination) For each $k > 1$, let $a_k = \mathrm{i}(Y^1) \cap \mathrm{i}(Y^{g_1}) \cap \mathrm{i}(Y^{g_k})$.

(Downstream Feasibility) If $|a_k| > \bar{q}_s^{g_k}$, then we add $\cup_{g \in \gamma(i)}(i, s, g) \cap Y^1$ to $R_s^{\rhd}(Y)$ such that $|\{j \in a_k | j \succ_s^{g_1} i\}| \geq \bar{q}_s^{g_k}$.

(Selection from Remaining) Let $\bar{Y}^1 = (Y^1 \cap Y^{g_1}) \setminus R_s^{\rhd}(Y)$. Let $C_s^{g_1}(Y) = \bar{Y} \subseteq \bar{Y}^1$ such that $|\bar{Y}| = \min\left\{\bar{Y}^1, \bar{q}_s^{g_1}\right\}$, and for each $(\bar{i}, i') \in \left(\mathrm{i}(\bar{Y}) \cap I^{g_1}\right) \times \left(\mathrm{i}(\bar{Y}^1) \cap I^{g_1} \setminus \mathrm{i}(\bar{Y})\right)$, $\bar{i} \succ_s^{g_1} i'$. Let $R_s^{g_1}(Y) = (Y^1 \cap Y^{g_1}) \setminus C_s^{g_1}(Y)$. Add $C_s^{g_1}(Y) \cup (\cup_{i \in \mathrm{i}(C_s^{g_1}(Y))}(Y_i \cap Y_s))$ to $C_s^{\rhd}(Y)$ and add $R_s^{g_1}(Y) \cup (\cup_{i \in \mathrm{i}(R_s^{g_1}(Y))}(Y_i \cap Y_s))$ to $R_s^{\rhd}(Y)$.

(Update Remaining Students and Capacities) Let $Y^2 = Y^1 \setminus (C_s^{\rhd}(Y) \cup R_s^{\rhd}(Y))$ and $\bar{q}_s^{g_k} = q_s^{g_k} - |C_s^{\rhd}(Y) \cap Y^{g_k}|$ for all $k > 1$.

In general;

Step $\bar{k}$: Grade $g_{\bar{k}}$ Selection

(Type Determination) For each $k > \bar{k}$, let $a_k = \mathrm{i}(Y^{\bar{k}}) \cap \mathrm{i}(Y^{g_{\bar{k}}}) \cap \mathrm{i}(Y^{g_k})$.

(Downstream Feasibility) If $|a_k| > \bar{q}_s^k$, then we add $\cup_{g \in \gamma(i)}(i, s, g) \cap Y^{\bar{k}}$ to $R_s^{\rhd}(Y)$ such that $|\{j \in a_k | j \succ_s i\}| \geq \bar{q}_s^{g_k}$.

(Selection from Remaining) Let $\bar{Y}^{\bar{k}} = \left(Y^{\bar{k}} \cap Y^{g_{\bar{k}}}\right) \setminus R_s^{\rhd}(Y)$. Let $C_s^{g_{\bar{k}}}(Y)$ be the $\bar{Y} \subseteq \bar{Y}^{\bar{k}}$ such that $|\bar{Y}| = \min\left\{\bar{Y}^{\bar{k}}, \bar{q}_s^{g_{\bar{k}}}\right\}$, and for each $(\bar{i}, i') \in \left(\mathrm{i}(\bar{Y}) \cap I^{g_{\bar{k}}}\right) \times \left(I^{g_{\bar{k}}} \setminus \mathrm{i}(\bar{Y})\right)$, $\bar{i} \succ_s^{g_{\bar{k}}} i'$. Let $R_s^{g_{\bar{k}}}(Y) = (Y^{\bar{k}} \cap Y^{g_{\bar{k}}}) \setminus C_s^{g_{\bar{k}}}(Y)$. Add $C_s^{g_{\bar{k}}}(Y) \cup (\cup_{i \in \mathrm{i}(C_s^{g_{\bar{k}}}(Y))}(Y_i \cap Y_s))$ to $C_s^{\rhd}(Y)$ and add $R_s^{g_{\bar{k}}}(Y) \cup (\cup_{i \in \mathrm{i}(R_s^{g_{\bar{k}}}(Y))}(Y_i \cap Y_s))$ to $R_s^{\rhd}(Y)$.

(Update Remaining Students and Capacities) Let $Y^{\bar{k}+1} = Y^{\bar{k}} \setminus (C_s^{\rhd}(Y) \cup R_s^{\rhd}(Y))$ and $\bar{q}_s^{g_k} = q_s^{g_k} - |C_s^{\rhd}(Y) \cap Y^{g_k}|$ for all $k > \bar{k}$.

The process concludes after Step $|G|$, and we arrive at $C_s^{\triangleright}(Y)$.

Proof of Proposition 2. We start with LAD. Consider any subset of contracts $Y \subseteq X$. By the definition, if $C_i(Y) \neq \emptyset$, then $\mathrm{s}_i(Y) \neq \emptyset$ and $|C_i(Y)| = |\gamma(i)|$. Hence, for any $Y \subset Y'$ $\mathrm{s}_i(Y') \neq \emptyset$ and $|C_i(Y')| = |\gamma(i)|$. That is, the parents' choice functions satisfy LAD.

Next, we show that C_i is not substitutable via example. Let i be a parent with $s\, P_i\, s'$ and $\gamma(i) = \{1, 2\}$. Let $Y = \{(i, s, 1), (i, s', 1), (i, s', 2)\}$ and $Y' = Y \cup \{(i, s, 2)\}$. Then, $C_i(Y) = \{(i, s', 1), (i, s', 2)\}$ and $C_i(Y') = \{(i, s, 1), (i, s, 2)\}$. Hence, parent i's choice function is not substitutable. ■

Proof of Proposition 3. We prove by example. Let $G = \{1, 2\}$. Let $I^1 = \{i, k\}$ and $I^2 = \{i.j\}$. Let $k \succ_s^1 i$ and $i \succ_s^2 j$. For each $g \in \{1, 2\}$ $q_s^g = 1$. Let $1 \triangleright 2$. Let $Y = \{(i, s, 1), (i, s, 2)\}$, $Y' = Y \cup \{(k, s, 1)\}$, $Y'' = Y \cup \{(j, s, 2)\}$, and $Y''' = Y'' \cup \{(k, s, 1)\}$. Then $C_s^{\triangleright}(Y) = \{(i, s, 1), (i, s, 2)\}$, $C_s^{\triangleright}(Y') = \{(k, s, 1)\}$, $C_s^{\triangleright}(Y'') = \{(i, s, 1), (i, s, 2)\}$ and $C_s^{\triangleright}(Y''') = \{(k, s, 1), (j, s, 2)\}$. Since $Y \subset Y'$ and $|C_s^{\triangleright}(Y)| > |C_s^{\triangleright}(Y')|$, $C_s^{\triangleright}$ does not satisfy LAD. Since $Y'' \subset Y'''$, $(j, s, 2) \notin C_s^{\triangleright}(Y'')$ and $(j, s, 2) \in C_s^{\triangleright}(Y''')$, C_s is not substitutable. ■

Proof of Proposition 4. We start with LAD. Let $\bar{I} \subset \bar{J} \subseteq I^g$. We compare $D_s^{g,\hat{q}}(\bar{I})$ and $D_s^{g,\hat{q}}(\bar{J})$. We consider the parents in $\bar{I}$ and $\bar{J}$ who are not rejected in Step 1 of $D_s^{g,\hat{q}}$. By definition, the number of parents who are not rejected in $\bar{J} \cap I^{g'}$ is weakly more than $\bar{I} \cap I^{g'}$ for all $g' \in G \setminus \{g\}$. Hence, the number of parents considered in Step 2 is weakly more when we consider $\bar{J}$ compared to $\bar{I}$. Then, $|D_s^{g,\hat{q}}(\bar{I})| \leq |D_s^{g,\hat{q}}(\bar{J})|$.

Next, we show subsitutability. Consider any subset of students $\bar{I} \subset I^g$. Let $i \notin D_s^{g,\hat{q}}(\bar{I})$. Then, when $D_s^{g,\hat{q}}(\bar{I})$ is considered i is rejected in either Step 1 or Step 2. Suppose i is rejected in Step 1. Then, we consider $D_s^{s,\hat{q}}(\bar{I} \cup \{j\})$ where $j \notin \bar{I}$. By definition, there are at least $\hat{q}^{g'}$ parents in $(\bar{I} \cup \{j\}) \cap I^{g'}$ with higher priority than $i \in I^{g'}$ according to $\succ_s^g$. Hence, $i \notin D_s^{s,\hat{q}}(\bar{I} \cup \{j\})$. Now, suppose i is rejected in Step 2. Then, all parents in $D_s^{s,\hat{q}}(\bar{I})$ have higher priority than i. By LAD and the definition, there will be at least $\hat{q}^g$ parents in Step 2 with higher priority than i when we consider $\bar{I} \cup \{j\}$. Hence, $i \notin D_s^{s,\hat{q}}(\bar{I} \cup \{j\})$. ■

Proof of Theorem 3. Let $g_1 \triangleright g_2 \triangleright \cdots \triangleright g_n$. First, observe that in the many-to-one subproblem consisting of I^{g_1}, by substitutability of each school's choice function $D_s^{g_1,q}$ and Hatfield and Milgrom (2005), the assignment μ^{g_1} is not blocked by any contract set that involves parents only in $\hat{I}^{g_1}$. $SCO^{\triangleright}$ is clearly *individually rational.*

Stability: Suppose by contradiction that there is $Z \subseteq X$ such that Z blocks $SCO^{\triangleright}$. Without loss of generality, let the $\triangleright$-earliest grade in $\mathrm{g}(Z)$ be g_1. Since for each $s \in S$, $C_s^{\triangleright}$ chooses grade 1 contracts (and siblings) first, $Z' = Z^{g_1} \cup \{(i,s,g) \in Z : i \in \mathrm{i}(Z^{g_1})\}$ is chosen in Step 1 for $C_s^{\triangleright}$. So Z' also blocks $SCO^{\triangleright}$.

This implies that in the many-to-one subproblem at grade 1 $\mathrm{i}(Z')$ blocks μ^{g_1}—a contradiction.

Strategy-proofness: For each $g \in G$, let $\hat{I}^g = \{i \in I : \ \gamma(i) = g\}$. Observe that each parent i is only involved in the construction of $SCO^{\triangleright}$ at her earliest grade $\gamma(i)$.

At the processing of grade $\gamma(i)$, i's report does not affect the assignment of $\mu^g, \ldots, \mu^{\gamma(i)-1}$—$i$ can only affect $\mu^{\gamma(i)}$. By substitutability and LAD of each school's choice function $D_s^{g,q}$ and Hatfield and Milgrom (2005), the cumulative offer process mechanism is *strategy-proof* and i cannot benefit by misreporting. ■