Which School Assignments are Legal?

Thayer Morrill*

September, 2016

Abstract

One rationalization for making a school assignment that eliminates justified envy is that otherwise a school district might be vulnerable to a lawsuit. However, in order for a plaintiff to have legal standing she must demonstrate that the actions caused her harm *and* that this harm is redressable. We define a set of assignments to be legal if whenever a student is harmed (has justified envy) there is no legal assignment where she is assigned to that school (her harm is not redressable). We show that for any school assignment problem, there is a unique set of legal assignments; the set of legal assignment is a superset of the assignments that eliminate justified envy; but the two sets have the same mathematical structure. Specifically, the Lattice Theorem, Decomposition Lemma, and Rural Hospital Theorem all hold. Moreover, there is a unique, Pareto efficient, legal assignment: the assignment made by Kesten's Efficiency Adjusted Deferred Acceptance mechanism when all students consent. Finally, we interpret legality in terms of fairness: a legal assignment is an assignment where any objection is either unjustified or petty.

School assignment programs have considered a number of objectives such as fairness, efficiency, strategic simplicity, and diversity. A critical consideration that has not been formally considered is which of these assignments are legal. Legality has sometimes been used as a justification for implementing an assignment with no justified envy[1] (for example, in Abdulkadiroglu et al, 2005).

*North Carolina State University. Email address: thayer_morrill@ncsu.edu. An early version of this paper was titled "Petty Envy When Assigning Schools". I thank Antonio Romero-Medina for the many conversations and espressos. Both were very enjoyable and productive for this work.

[1]Student i has **justified envy** of school a under an assignment if i prefers a to her assignment and i has higher priority at a than one of the students assigned to a.

The presumption has been that no student could win a lawsuit against the school district if her priorities have not been violated.

However, the legality of an assignment with no justified envy does not imply the illegality of an assignment that violates a student's priority. Legal standing, or *locus standi*, is the capacity to bring suit in court. As interpreted by the United States Supreme Court:

> Under modern standing law, a private plaintiff seeking to bring suit in federal court must demonstrate that he has suffered "injury in fact," that the injury is "fairly traceable" to the actions of the defendant, and that injury will *"likely be redressed by a favorable decision."*[2]

In order for a student to have legal standing she must both have been harmed and this harm must be redressable. The literature has considered the first condition but not the second. A student i has been "harmed" by assignment μ if i would prefer an alternative school a to her assignment under μ and i has higher priority than one of the students assigned to a. But this harm is only "redressable" if there is a legal assignment under which i is assigned to a.

At first glance, this definition appears circular. Whether or not an assignment violates a student's rights, and consequently is illegal, depends on which assignments are legal. However, we will show that this definition is iterative and not circular. Assignments where no student has justified envy must be legal since no student has been harmed. Having determined that some assignments are legal, we can now determine that some assignments are illegal; specifically, an assignment is illegal if it violates the priority of a student i at a school a and there exists an assignment with no justified envy in which i is assigned to a. Such a student has been harmed and this harm is redressable.

Having determined that some assignments are illegal, we can now determine that additional assignments are legal. Suppose there exists a student i and a school a such that every assignment in which i is assigned to a has been determined to be illegal. Now consider an assignment where i has justified envy of school a. The key point is that i would not have legal standing to file a lawsuit against such an assignment. She has been harmed by not being assigned to a, but since any assignment in which she receives a is illegal, this harm is not redressable.

[2] This quote is from Hessick (2007) regarding Supreme Court case Lujan vs. Defenders of Wildlife, 504 U.S. 555, 560-61 (1992).

This defines an iterative process to determine which assignments are legal. As more assignments are determined to be legal, we are able to determine that more assignments are illegal. But when we discover that more assignments are illegal, we discover more priorities that do not need to be respected. Consequently, we find more assignments that are legal. Determining that additional assignments are legal implies that more assignments must be illegal. Determining that additional assignments are illegal implies that more assignments are legal, and so on.

Our main result is that at the conclusion of this iterative process, every assignment has been determined to be either legal or illegal. In general, the set of legal assignments is a superset of the set of assignments with no justified envy. However, for a legal assignment with justified envy, our process allows us to articulate to a decision maker (or judge) precisely why this priority does not need to be respected but other priorities do.

We show that the set of legal assignments has the same structure as the set of stable assignment in the classic marriage problem. In particular, it is a sublattice. Therefore, there is a well-defined, unique, student-optimal fair assignment. We show a surprising result: this assignment is Pareto efficient and corresponds to the assignment made by Kesten's Efficiency Adjusted Deferred Acceptance algorithm (Kesten 2010, hereafter EADA).[3]

Legality would typically be considered a constraint on the market design and not necessarily an objective in and of itself. Fairness and efficiency are the primary design objectives of economists. We next consider whether or not legal assignments are fair. Typically, the literature calls an assignment unfair a student i has justified envy of school a. But what if we could determine that it is impossible to assign i to a? Is it fair to not assign j to a just to keep i from being jealous? Typically, people would not call this fair and instead would call it petty.

Under this alternative interpretation of fairness, an assignment is unfair if a student i wants a school a (i prefers a to her assignment), i deserves a (i has higher priority at a than one of the students assigned to a), *and* it is possible to assign i to a (there exists a fair assignment where i is assigned to a). We say i has petty envy if i desires a but it is impossible to assign i to a.

We demonstrate that this interpretation of fairness is equivalent to legality. A central tension in school assignment is that it is impossible for a mechanism to eliminate justified envy and be Pareto

[3]More precisely, the assignment made by the efficiency adjusted deferred acceptance algorithm when all student's consent.

efficient. There is no tension under our interpretation of fairness. There always exists a unique assignment that is fair and efficient: Kesten's EADA assignment. This is the main conclusion of our paper. Not only is it possible to make a fair and efficient assignment, but this assignment would not be vulnerable to a lawsuit.[4]

1 Relationship to the Literature

To the best of our knowledge, ours is the first paper to formally consider the legality of school assignments. However, there are a number of papers that have considered alternative interpretations of fairness. Kesten (2004) defines an assignment to be reasonably fair if whenever student i has justified envy of a school a, then there is no fair assignment that assigns i to a. Kesten introduces several algorithms, including EADA, that Pareto improve the DA assignment and are reasonably fair. Alcalde and Romero (2015) consider a fairness notion closely related to reasonable fairness. They allow a student i to school to school a if there is an assignment with no justified envy in which i is assigned to a.[5] They call unblocked assignments α-equitable and show that an assignment is α-equitable if and only if it weakly Pareto dominates an assignment with no justified envy. While α-equity is similar in spirit to our definition, there are important differences. Most importantly, the set of α-equitable assignments is unfair (using our definition) in the sense that one α-equitable assignment may block another. This can be seen in our Example 2. The fairness concept that is closest to ours is essentially stable introduced by Kloosterman and Troyan (2016). We discuss this paper in Section 6 after formally introducing our notion of pettiness.

Our notion of petty envy is in the spirit of bargaining sets introduced by Zhou (1994). For a

[4] We do not discuss in this paper whether or not it is possible to implement the Kesten assignment. There is reason to be pessimistic on this front. Kesten 2010 has a full discussion on the topic, but he proves that it is impossible for a strategyproof mechanism to implement his assignment. Dur and Morrill (2016) prove that no manipulatable assignment implements the Kesten assignment even in a Nash equilibrium.

[5] This is not the way α-equity is defined in Alcalde and Romero (2015), but it is equivalent to their definition. Specifically, they define (i, μ') to be an ϵ-objection to μ if $\mu'_i P_i \mu_i$ and $i \succ_{\mu'_i} j$ where $\mu_j = \mu'_i$. An ϵ-objection (i, μ') is admissible if no student has an ϵ-objection to μ'. An assignment μ is α-equitable if there is no admissible ϵ-objection to μ. An ϵ-objection is equivalent to an assignment being blocked by another assignment. Since any assignment with justified envy can be blocked, only an assignment with no justified envy has no counter objections. Therefore, an assignment is α-equitable if it is not blocked by any assignment that eliminates justified envy. Reasonable stability, introduced by Cantala and Papai (2014), is closely related to α-equity.

transferable utility, cooperative game, he introduces bargaining sets as a generalization of the core. Specifically, a coalition can block with an alternative only if that alternative is not subsequently blocked. He demonstrates that the bargaining set is non-empty for every transferable-utility game. Most closely related to our paper is the literature on von Neumann-Morgenstern Stable sets (hereafter, vNM stable sets). We show in Section 6 that for the school assignment problem, legality is equivalent to vNM stability. Therefore, our paper is closely related to the work of Ehlers (2007), Wako (2008), Wako (2010), and Bando (2014) who study vNM stability for the marriage problem. We discuss the relationship between our work in these papers when we demonstrate the connection to vNM stability in Section 6.

Our approach can be viewed as dividing priorities into those that must be honored and those that can be disregarded. A paper that considers a similar notion is Dur, Gitmez, and Yilmaz (2015). Intuitively, they define an assignment to be partially fair if the only priorities that are violated are "acceptable violations". The key difference between petty envy and partial fairness is that the priority violations deemed petty are endogenous to our model whereas the set of "acceptable violations" for partial fairness is exogenously given. Dur et al. (2015) also provide a striking result regarding EADA. They show that the unique mechanism that is constrained efficient, no-consent proof[6], and Pareto improves the DA assignment is EADA. We show that EADA is the unique efficient and legal assignment. Their conditions are properties of a mechanism while our conditions are properties of an assignment, so our results do not relate to each other directly; nonetheless, there is clearly a strong and complementary relationship between the two results.

2 Model

There is a set of students, $A = \{i, j, k, \ldots\}$, to be assigned to a set of schools, $O = \{a, b, c, \ldots\}$. Each student i has strict preferences P_i over the schools. We allow students to be unassigned. iP_ia indicates that student i prefers being unassigned to being assigned to school a. Each school a has priorities $\succ_a$ over the students. R and $\succeq$ represent the weak preferences and priorities corresponding to P and $\succ$, respectively. Each school has a maximum number of students that it can be assigned. We let q_a denote the maximum capacity of school a. For simplicity, we assume that every student

[6]See Dur et al. (2015) for a formal definition of no-consent proof, but intuitively a mechanism is no-consent proof if no student is harmed by allowing her priority at a given object to be violated.

is acceptable to every school although this assumption is not necessary for any of our results to hold.

An assignment μ is a function from students to schools such that no school a is assigned more than q_a students. $\mu_i = a$ indicates that student i is assigned to school a. In a slight abuse of notation, for each school a we set $\mu_a = \{i : \mu_i = a\}$. $\mu_i = i$ indicates that student i is left unassigned.

An assignment μ is **individually rational** if for every student i $\mu_i R_i i$. An assignment μ is **wasteful** if there is a student i and a school a such that $a P_i \mu_i$ and $|\mu_a| < q_a$.

3 Legal Assignments

We seek to determine the legality of every potential assignment. An assignment is only illegal if a student has been harmed and this harm is redressable. First, if an assignment is wasteful or not individually rational, then we interpret that assignment as being illegal since it is trivial to address these objections. For expositional convenience, we focus on the non-trivial case when an assignment is both individually rational and non-wasteful. Given an assignment μ, we interpret i as being harmed by μ if i has justified envy of some school a under μ. We interpret i's harm as redressable if there exists a legal assignment where i is assigned to a.

Definition 1. L is a **legal set of assignments** if $\mu \notin L$ (μ is illegal) if and only if there exists a student i and a school a such that

1. $a\ P_i\ \mu_i$ and for some student $j \in \mu_a$, $i \succ_a j$ (i has been harmed)
2. there exists a $\nu \in L$ such that $\nu_i = a$ (i's harm is redessable)

It is not obvious that a legal set of assignments exists. Moreover, if a legal set of assignments existed but was not always unique, then our definition of legality would be ambiguous. Our main theorem is to demonstrate that a legal set of assignments always exists and is unique.

Theorem 1. *There exists a unique legal set of assignments.*

We prove Theorem 1 constructively. Central to our approach will be the notion of a student blocking an assignment with an alternative assignment.

Definition 2. Student i **blocks** assignment μ with assignment ν if for some school a: (1) $aP_i\mu_i$; (2) $i \succ_a j$ where $\mu_j = a$; and (3) $\nu_i = a$. We also say μ is blocked by ν if there is a student i who blocks μ with ν.

We will use the following function repeatedly, so we define it explicitly. Given any set of assignments S, $\pi(S)$ is the set of assignments that are not blocked by any assignment in S.

$$\pi(S) = \{\mu | \nexists \nu \in S \text{ such that } \nu \text{ blocks } \mu\} \tag{1}$$

The following two facts are useful:

Monotonicity Fact: If $A \subseteq B$, then $\pi(B) \subseteq \pi(A)$.

Justified Envy Fact: If J are the assignments with no justified envy, then $J \subseteq \pi(A)$ for any set of assignments A.

Our objective is to find a legal set of assignments. An assignment with no justified envy is not blocked by any assignment, legal or otherwise. Consequently, any legal set of assignments must contain all assignments with no justified envy. Therefore, we start with this set. We refer to this set as "too small" as we anticipate finding assignments that are only blocked by illegal assignments and therefore are legal.

$$S^1 := \text{assignments with no justified envy.}$$

We are defining an iterative process. We next define:

$$B^1 = \pi(S^1)$$

The assignments that eliminate justified envy (S^1) are legal. An assignment blocked by a legal assignment is illegal. Therefore, each assignment $\mu \notin B^1$ must be illegal. However, we anticipate that B^1 is likely "too big" as we expect to find that some of the assignments in B^1 are illegal.

In general, for any $k > 1$ we define the small and big sets as follows:

$$S^k = \pi(B^{k-1}) \tag{2}$$

$$B^k = \pi(S^k) \tag{3}$$

The assignments not in B^1 have been determined to be illegal. Consider the set $S^2 = \pi(B^1)$ and let $\mu \in S^2$. If $\mu \in S^1$, then μ is legal. If $\mu \in S^2 \setminus S^1$, then every assignment that blocks μ is illegal (specifically, if ν blocks μ, then $\nu \notin B^1$ by definition). Since μ is never blocked by a legal assignment, μ must be legal. Therefore, S^2 must be contained in any legal set of assignments. Similarly, any assignment $\mu \notin B^2$ is illegal, and so on. A key point is that the number of assignments we have determined to be legal is weakly increasing.

Lemma 1. *For every $k > 1$, $S^{k-1} \subseteq S^k$.*

Proof. Our proof is by induction. $S^1 \subseteq S^2$ by the No Justified Envy Fact. Now consider a $k > 1$. By the inductive hypothesis, $S^{k-2} \subseteq S^{k-1}$. Therefore, by the Monotonicity Fact, $\pi(S^{k-1}) = B^{k-1} \subseteq \pi(S^{k-2}) = B^{k-2}$. Since $B^{k-1} \subseteq B^{k-2}$, again by the Monotonicity Fact, $\pi(B^{k-2}) = S^{k-1} \subseteq \pi(B^{k-1}) = S^k$. □

As there are only a finite number of possible assignments, eventually the process must terminate. Let n be the first integer such that $S^n = S^{n+1}$. Our next theorem establishes that at the conclusion of the iterative process, every assignment has been determined to be either legal or illegal.

Theorem 2. *$S^n = B^n$.*

We prove Theorem 2 in the Appendix. However, we can immediately conclude that S^n is the unique legal set of assignments.

Proof. (of Theorem 1) By Theorem 2, $S^n = B^n = \pi(S^n)$. Therefore, it is straightforward to verify that $\mu \notin S^n$ if and only if μ is blocked by a $\nu \in S^n$. Consequently, S^n is a legal set of assignments.

To show uniqueness, suppose A is a legal set of assignments. We first show that $A = \pi(A)$. No assignment $\mu \in A$ blocks an assignment $\nu \in A$ or else ν would be illegal. Therefore, $A \subseteq \pi(A)$. Consider $\mu \in \pi(A)$. By the definition of $\pi(\cdot)$, no assignment in A blocks μ. By the definition of legality, since no assignment in A blocks μ, $\mu \in A$. Therefore $\pi(A) \subseteq A$, and indeed, $A = \pi(A)$.

An assignment with no justified envy must be legal, therefore, $S^1 \subseteq A$. By the Monotonicity Fact, $\pi(A) \subseteq \pi(S^1)$. Since $\pi(A) = A$ and $\pi(S^1) = B^1$, $A \subseteq B^1$. Again, by the monotonicity fact, $\pi(B^1) \subseteq \pi(A)$ implying $S^2 \subseteq A$. Iterating this argument, we get that for any k, $S^k \subseteq A \subseteq B^k$. Since $S^n = B^n$, it must by that $S^n = A$. $\square$

4 Lattice Structure

The set of legal assignments is a superset of the set assignments that eliminate justified envy. But we show that the two sets have the same mathematical structure. Specifically, the Lattice Theorem, Decomposition Lemma, and Rural Hospital Theorem all hold. To emphasize the connection with classical matching theory, our Lemmas and proofs mirror the presentation in Roth and Sotomayor (1990).

Our process is a modification of the "cloning" procedure use by Gale and Shapley (1962) and covered extensively in Roth and Sotomayor (1990). If school a has a capacity of q, then we create q **seats** at a, $a^1, \ldots, a^q$. Each seat at a has the same priorities over students as the school a. Unlike the cloning procedure, we do not define any student preferences over seats. A student only has preferences over schools. A school assignment assigns each student to a school. A **seat assignment** assigns each student to a seat at a school. Given a school assignment μ, a seat assignment $\overline{\mu}$ is a **μ-seat assignment** if for each student i, $\overline{\mu}_i$ is a seat at μ_i. For expositional ease, the seat assignment $\overline{\mu}$ is understood to be a μ-seat assignment.

Definition 3. Given two school assignments μ and ν, $\overline{\mu}$ and $\overline{\nu}$ are **consistent seat assignments** if for any student i such that $\mu_i = \nu_i$, $\overline{\mu}_i = \overline{\nu}_i$.

In words, if i receives the same assignment under μ and ν, then i is assigned to the same seat under $\overline{\mu}$ and $\overline{\nu}$. We will work exclusively with consistent seat assignments. Given two seat assignments, $\overline{\mu}$ and $\overline{\nu}$, we induce a graph $G^{\overline{\mu},\overline{\nu}}$ as follows (when $\overline{\mu}$ and $\overline{\nu}$ are clear from context, we will refer to the graph as G). Each student and each seat is a vertex. There is an edge between student i and seat s if i is assigned to s under either $\overline{\mu}$ or $\overline{\nu}$.[7] Each vertex has degree less than or equal to two; therefore, each connected component of the graph is either a path or a cycle.

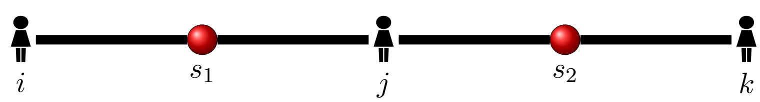
[7] If $\mu_i = \nu_i = i$, then we draw an edge from i to itself. If i is assigned to s under both $\overline{\mu}$ and $\overline{\nu}$ then we draw two edges between i and s. We refer to both as trivial cycles.

Now, suppose that μ and ν are individually rational, nonwasteful, and do not block each other. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments and let $G = G^{\overline{\mu},\overline{\nu}}$. We will define **pointing** analogously to the discussion in Roth and Sotomayor (1990). We ask each student i to point at the seat of her favorite school (between μ_i and ν_i). Similarly, if a seat is assigned to different students under μ and ν, then we ask her to point at the student with highest priority.

Lemma 2 (Pointing Lemma)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other, and let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. Then no two students point to the same seat. Moreover, a student i and seat only point at each other if $\mu_i = \nu_i$.*

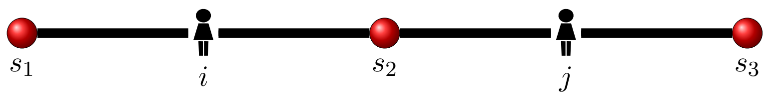
Proof. Suppose for contradiction that both i and j point at seat s where s is a seat at school a. Without loss of generality, $i \succ_a j$ and $\overline{\mu}_i = s$ (and consequently that $\overline{\nu}_j = s$). Therefore, $\mu_i = a$ and $\nu_j = a$. Note that $\mu_i P_i \nu_i$, $i \succ_{\mu_i} j$, and $\nu_j = \mu_i$. Therefore, i blocks ν with μ, a contradiction. Similarly, suppose i points at seat s and seat s points to i but $\mu_i \neq \nu_i$. Let s be a seat at school a (i.e. $\mu_i = a$). i prefers μ_i to ν_i. a must be assigned to capacity by ν or else ν would be wasteful. Let j be the student assigned to seat s. Since s points at i, $i \succ_a j$. But since $aP_i\nu_i$ and $i \succ_a j$ where $\nu_j = a$, i blocks ν with μ, a contradiction. □

Potentially, a path between students and seats could occur three different ways: each endpoint of the path is a student, each endpoint is a school, or else one endpoint is a student and one endpoint is a school. Consider the path below where each endpoint is a student (more generally, a path $\{i_1, s_1, i_2, s_2, \ldots, s_{n-1}, i_n\}$).



Student i must point at seat s_1 (otherwise, i prefers being unassigned to s_1 and the assignment of i to s_1 is not individually rational). Similarly, k must point at s^2. Therefore, whichever seat j points to has two students pointing at it, contradicting the Pointing Lemma. More generally, n students, $\{i_1, \ldots, i_n\}$, point at $n - 1$ seats, $\{s_1, \ldots, s_{n-1}\}$. By the pigeon hole principle, two students must be pointing at the same seat which is a contradiction.

Suppose instead that the endpoints are seats such as the path below (more generally, a path $\{s_1, i_1, s_2, i_2, \ldots, i_{n-1}, s_n\}$).



Note that when i is assigned to s_2, s_1 is unassigned. Therefore, i cannot point at s_1 or else it would be wasteful to leave s_1 unassigned. Similarly, j cannot point at s_3. Therefore, both i and j point at s_2, a contradiction. More generally, the $n-1$ students, $\{i_1, \ldots, i_{n-1}\}$, point to $n-2$ schools $\{s_2, \ldots, s_{n-1}\}$ implying that two students must point at the same school, a contradiction.

Finally, consider the path below where one endpoint is a student and one endpoint is a school (more generally (more generally, a path $\{i_1, s_1, i_2, \ldots, i_n, s_n\}$).

Repeating the logic of the previous two cases, i must point at s_1 or else her assignment to s_1 would not be individually rational. j cannot point to s_2 or else it would be wasteful to leave s_2 unassigned. But this implies that i and j both point at s_1, a contradiction. More generally, since n students, $\{i_1, \ldots, i_n\}$ point at $n-1$ schools, $\{s_1, \ldots, s_{n-1}\}$, two students must point at the same seat which is a contradiction.

Therefore, we conclude that there are no paths in our graph. Each partition of students and seats must be a cycle. We summarize this in the following lemma.

Lemma 3. *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. Moreover, let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. Then each student i_1 is part of a unique cycle $\{i_1, s_1, \ldots, i_n, s_n\}$ where $\overline{\mu}_{i_k} = s_k = \overline{\nu}_{i_{k+1}}$.*[8]

We will refer to $\{i_1, s_1, \ldots, i_n, s_n\}$ as i_1's $(\overline{\mu}, \overline{\nu})$**-cycle**. Note that different seat assignments produce potentially different cycles. We give an example to aid with intuition.

Example 1. Let there be three schools, a, b, and c, and eight students $\{i_1, i_1, \ldots, i_8\}$. Schools a and b have a capacity of 2 and school c has a capacity of 3. Define assignments μ and ν by

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & a & a & b & b & b & c & c \end{pmatrix} \quad \nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & b & c & a & b & c & a & b \end{pmatrix}$$

[8]It is understood that $i_{n+1} = i_1$. As a reminder, if $\mu_i = \nu_i$, then we have constructed i to be part of a trivial cycle.

Note that i_1 and i_5 are the only students that receive the same assignment under μ and ν. Therefore, our only restriction on a consistent seat assignment is that i_1 and i_5 must each be assigned to the same seat. In the table below, $\overline{\mu}$ and $\overline{\nu}$ are consistent seat assignments. $\overline{\mu}$ and $\overline{\nu}'$ are also consistent seat assignments. However, $\overline{\mu}$ and $\overline{\nu}''$ are not consistent seat assignments because i_5 is assigned to a different seat at b under $\overline{\mu}$ and $\overline{\nu}''$.

$\overline{\mu}$		$\overline{\nu}$	$\overline{\nu}'$	$\overline{\nu}''$	$\overline{\mu}$		$\overline{\nu}$	$\overline{\nu}'$	$\overline{\nu}''$	$\overline{\mu}$		$\overline{\nu}$	$\overline{\nu}'$	$\overline{\nu}''$
i_1	a^1	i_1	i_1	i_1	i_4	b^1	i_2	i_8	i_5	i_7	c^1	i_3	i_3	i_3
i_2	a^2	i_4	i_7	i_7	i_5	b^2	i_5	i_5	i_8	i_8	c^2	i_6	i_6	i_6
i_3	a^3	i_7	i_4	i_4	i_6	b^3	i_8	i_2	i_2					

In this example, the $(\overline{\mu}, \overline{\nu})$-cycles are (i_1, a^1), (i_5, b^2), (i_2, a^2, i_4, b^1), (i_3, a^3, i_7, c^1), and (i_6, b^3, i_8, c^2). The $(\overline{\mu}, \overline{\nu}')$-cycles are (i_1, a^1), (i_5, b^2), and $(i_2, a^2, i_7, c^1, i_3, a^3, i_4, b^1, i_8, c^2, i_6, b^3)$.

An immediate consequence of Lemma 3 is our version of the Rural Hospital Theorem.

Lemma 4 (Rural Hospital)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. A student i is assigned under μ if and only if i is assigned under ν. Moreover, every school is assigned the same number of students under μ and ν.*

Proof. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. By Lemma 3, every student who receives an assignment is part of a unique cycle. This implies that every student and every seat is either assigned under both μ and ν or under neither. $\square$

The Pointing Lemma tells us that a student and a seat cannot point at each other. This implies our version of the Decomposition Lemma. Consider any student i_1 who is part of a nontrivial cycle $\{i_1, s_1, \ldots, i_n, s_n\}$. If i_1 points at s_1, then s_1 must point at i_2. But if s_1 points at i_2, then i_2 must point to s_2, and so on. Similarly, if i_1 points at s_n, then s_n must point at i_n who must point at s_{n-1} and so on. The key point is that after learning the preferences of one student i, we can infer both the preferences of every student and the priorities of every seat in i's cycle. Similarly, learning for any one seat which student has higher priority implies the preferences and priorities of all other students in the cycle.

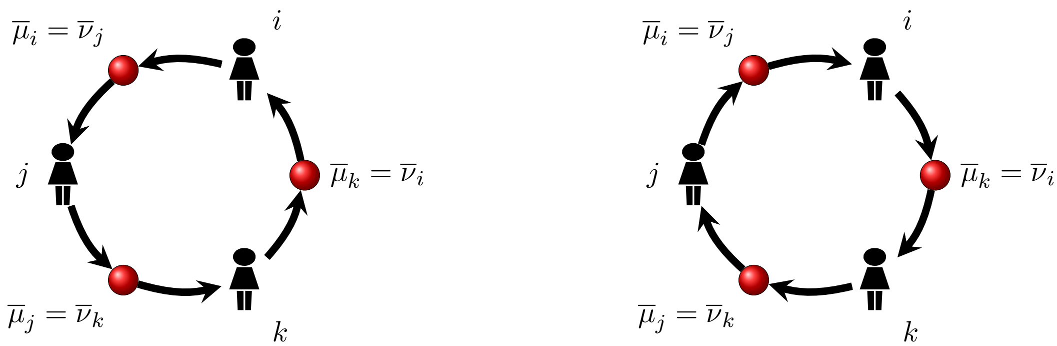


Figure 1: Every cycle must have a well-defined orientation. Either all students prefer μ and all schools "prefer" ν or else the opposite.

Lemma 5 (Decomposition Lemma)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. Moreover, let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments and let $\{i_1, s_1, \ldots, i_n, s_n\}$ be a nontrivial $(\overline{\mu}, \overline{\nu})$-cycle. Then either*

$$s_k \; P_{i_k} \; s_{k-1} \text{ and } i_{k+1} \succ_{s_k} i_k \text{ for every } 1 \leq k \leq n$$

or else

$$s_{k-1} \; P_{i_k} \; s_k \text{ and } i_{k-1} \succ_{s_{k-1}} i_k \text{ for every } 1 \leq k \leq n.$$

Note that we can now prove a strong version of the Rural Hospital Theorem (which is true for the college admissions problem as well). The Rural Hospital Theorem says that a school is assigned to the same number of students under any two individually rational, nonwasteful assignments that do not block each other. In fact, if a school ever is assigned to less than its capacity, then it receives exactly the same set of students under either assignment.

Lemma 6 (Strong Rural Hospital Theorem)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. For any school a, if $|\mu_a| < q_a$, then $\mu_a = \nu_a$.*

Proof. Suppose for contradiction that $|\mu_a| < q_a$ but that $\mu_a \neq \nu_a$. By the Rural Hospital Theorem, $|\mu_a| = |\nu_a|$, and in particular, $\mu_a \not\subseteq \nu_a$. Therefore, there exists a student i such that $\mu_i = a$ but $\nu_i \neq a$. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. i is part of a unique $(\overline{\mu}, \overline{\nu})$-cycle, and in particular there exists a student j who is assigned to i's seat at a by $\overline{\nu}$. Specifically, $\overline{\mu}_i = \overline{\nu}_j$. i cannot prefer μ to ν or else ν is wasteful (a is not assigned to its capacity under ν). However, by the Decomposition Lemma, all students in the cycle prefer ν to μ. Since $\nu_j = a$, μ is wasteful as $aP_j\mu_j$ and a has available seats under μ. $\square$

We have only specified priorities at a school for individual students. Therefore, in general we cannot compare two different sets of students. However, we will show that when two assignments do not block each other, we can compare the set of students assigned to a school.

Definition 4. Consider a school a and two sets of students A and B such that $|A| = |B|$ but $A \neq B$. Then

$$A >_a B \Leftrightarrow \text{ for all } i \in A \setminus B \text{ and } j \in B \setminus A,\ i \succ_a j$$

We say that $A \geq_a B$ if either $A = B$ or else $A >_a B$.

Lemma 7. *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. For any school a, if $\mu_a \neq \nu_a$, then either $\mu_a >_a \nu_a$ or else $\nu_a >_a \mu_a$.*

Proof. Label $\mu_a \setminus \nu_a = \{i_1, i_2, \ldots, i_n\}$ in decreasing order of priority (i.e. $i_k \succ_a i_{k+1}$). By the Rural Hospital Theorem, $|\nu_a \setminus \mu_a| = |\mu_a \setminus \nu_a|$. Label $\nu_a \setminus \mu_a = \{j_1, j_2, \ldots, j_n\}$ in decreasing order of priority. Without loss of generality, $i_n \succ_a j_n$. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments such that i_n and j_n are assigned to a's first seat, a^1 ($\overline{\mu}_{i_n} = \overline{\nu}_{j_n} = a^1$). Then i_n and j_n are in the same $(\overline{\mu}, \overline{\nu})$-cycle. By the General Decomposition Lemma, since $i_n \succ_a j_n$, $\nu_{i_n} P_{i_n} \mu_{i_n} = a$.

Now define a new seat assignment, $\overline{\nu}'$ by switching j_n and j_1's seats under $\overline{\nu}$. In particular, $\overline{\nu}'_{j_1} = a^1 = \overline{\mu}_{i_n}$. It is straightforward to verify that $\overline{\mu}$ and $\overline{\nu}'$ are consistent seat assignments. Now i_n and j_1 are in the same $(\overline{\mu}, \overline{\nu}')$-cycle, but i_n's seat assignments have not changed. Therefore, i_n still points at the same seat, $\overline{\nu}'_{i_n}$. By the General Decomposition Lemma, every student in the cycle points to $\overline{\nu}'$, and every seat must point to $\overline{\mu}$. In particular, $i_n \succ_{a^1} j_1$. This proves the desired result since i_n is the student in $\mu_a \setminus \nu_a$ with the lowest priority at a and j_1 was the student in $\nu_a \setminus \mu_a$ with the highest priority at a. $\square$

We are now ready to prove the Lattice Theorem. Given two assignments μ and ν, we define the ordering $\geq$ by

$$\mu \geq \nu \text{ if for every student } i,\ \mu_i R_i \nu_i \text{ and every school } a, \nu_a \geq_a \mu_a \tag{4}$$

We define $\mu > \nu$ if the preference is strict. Given two assignments μ and ν, define $\mu \vee \nu_i = \max_i \{\mu_i, \nu_i\}$. Define $\mu \wedge \nu_i = \min_i \{\mu_i, \nu_i\}$.

Lemma 8. *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. For any school a,*

$$\mu \vee \nu_a := \{i | \mu \vee \nu_i = a\} = \min_{\geq_a} \{\mu_a, \nu_a\}$$

and

$$\mu \wedge \nu_a := \{i | \mu \wedge \nu_i = a\} = \max_{\geq_a} \{\mu_a, \nu_a\}.$$

Proof. If $\mu_a = \nu_a$, then the result is trivial. Now, suppose that $\mu_a \neq \nu_a$. By Lemma 7, either $\mu_a >_a \nu_a$ or else $\nu_a >_a \mu_a$. Without loss of generality, $\mu_a >_a \nu_a$. Consider any $i \in \nu_a \setminus \mu_a$. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. i is part of a $(\overline{\mu}, \overline{\nu})$-cycle. Since $\mu_a >_a \nu_a$, $\overline{\nu}_i$ (i's seat at a under the seat assignment $\overline{\nu}$) does not point at i. Therefore, by the General Decomposition Lemma, i points to $\overline{\nu}_i$ and in particular $aP_i\mu_i$. Therefore, for any $i \in \nu_a \setminus \mu_a$, $\mu \vee \nu_i = a$ and $\mu \wedge \nu_i \neq a$. Now consider any $j \in \mu_a \setminus \nu_a$ (by the Rural Hospital Theorem, such a j exists). Since we know $j \succ_a i$, $\mu_i = a$, and $\nu_j \neq a$, it must be that $\nu_j P_j \mu_j = a$ or else j would block ν with μ. Therefore, for any $j \in \mu_a \setminus \nu_a$, $\mu \wedge \nu_i = a = \mu_i$ and $\mu \vee \nu_i \neq a$. So indeed, $\mu \vee \nu_a = \nu_a = \min_{\geq_a} \{\mu_a, \nu_a\}$ and $\mu \wedge \nu_a = \mu_a = \max_{\geq_a} \{\mu_a, \nu_a\}$. $\square$

We can immediately conclude from Lemma 8 that $\mu \vee \nu$ and $\mu \wedge \nu$ are proper assignments when μ and ν are individually rational and nonwasteful assignments that do not block each other. For any school a, by Lemma 7 either $\mu_a = \nu_a$, $\mu_a >_a \nu_a$, or else $\nu_a >_a \mu_a$. Therefore, $\min_{\geq_a}$ and $\max_{\geq_a}$ are well defined sets, and since μ and ν are proper assignments, no more than q_a students are assigned to any school a.

We now prove the Lattice Theorem. From the construction in Section 3, we defined S^1 to be the assignments with no justified envy, and for any k, $B^k = \pi(S^k)$ (the assignments not blocked by any assignment in S^k), and $S^{k+1} = \pi(B^k)$. We fixed an n such that $S^n = S^{n+1}$. For our purposes, the following two points are critical. First, if $\mu, \nu \in S^n$, then μ and ν do not block each other.[9] Second, if $\mu \notin S^n$, then there exists a $\nu \in B^n$ such that ν blocks μ.[10]

[9] By definition, B^n are the assignments that are not blocked by an assignment in S^n. Since $S^n \subseteq B^n$, no assignment in S^n blocks an assignment in S^n.

[10] By definition, S^{n+1} are the assignments not blocked by an assignment in B^n. Therefore, the result follows from the fact that $S^{n+1} = S^n$.

Theorem 3 (Lattice Theorem)**.** *Let $\mu, \nu \in S^n$. Then $\mu \vee \nu$ and $\mu \wedge \nu$ are in S^n.*

Proof. Let $\mu, \nu \in S^n$. It is straightforward to verify that both are individually rational and non-wasteful. By construction, they do not block each other. Let $\lambda = \mu \vee \nu$ and let $\tau = \mu \wedge \nu$. Suppose for contradiction that $\lambda \notin S^n$. Then λ is blocked by some student i and some assignment $\lambda' \in B^n$. Specifically, for some school a $\lambda'_i = aP_i\lambda_i$ and there exists a $j \in \lambda_a$ such that $i \succ_a j$. By Lemma 8, $\lambda_a = \min_{\geq_a} \{\mu_a, \nu_a\}$. Without loss of generality, assume $\lambda_a = \mu_a$. Since $\lambda_i = \max_i \{\mu_i, \nu_i\}$, $\lambda_i R_i \mu_i$. Therefore, $\lambda'_i P_i \mu_i$. Since $\lambda'_i = aP_i\mu_i$ and $i \succ_a j \in \mu_a$, i blocks μ with λ', a contradiction since $\mu \in \pi(B^n)$.

Similarly, suppose i blocks τ with $\tau' \in B^n$. Let $a = \tau'_i$. By definition, $aP_i\tau_i$, and there exists a $j \in \tau_a$ be such that $i \succ_a j$. Without loss of generality, assume $\tau_i = \mu_i$. Since $\tau_i \neq a$, $\tau_a \neq \mu_a$. By Lemma 8, $\tau_a = \nu_a$ and $\nu_a >_a \mu_a$. Therefore, either $j \in \mu_a$ or else $j \in \nu_a \setminus \mu_a$. If $j \in \mu_a$, then i blocks μ with τ' since $aP_i\mu_i$ and $i \succ_a j$. If $j \in \nu_a \setminus \mu_a$, then by the Rural Hospital Theorem, there exists a $k \in \mu_a \setminus \nu_a$. But since $\nu_a >_a \mu_a$, $j \succ_a k$. Therefore, $i \succ_a k$. Since $aP_i\mu_i$, and $i \succ_a k$ and $\mu_k = a$, i blocks μ with τ'. Therefore, in either case, i blocks μ with τ'. This is a contradiction since $\mu \in \pi(B^n)$. $\square$

There are a number of interesting conclusions that are an immediate consequence of the Lattice Theorem. We now know that there is a student optimal legal assignment (and a student pessimal). Moreover, we can conclude that there is at most one Pareto efficient legal assignment. Otherwise, if μ and ν are both Pareto efficient, and contained in a fair set of assignments, then $\mu \vee \nu$ would be a well defined assignment that Pareto dominated both. Later, we will demonstrate that there exists a Pareto efficient legal assignment.

Corollary 4. *There exists at most one Pareto efficient legal assignment.*

5 Relationship to the Efficiency Adjusted Deferred Acceptance Algorithm

Kesten (2010) introduces a new mechanism for the school assignment problem: the Efficiency Adjusted Deferred Acceptance Algorithm (hereafter EADAM). Kesten identifies the source of DA's inefficiency, called interrupter students, and resolves this inefficiency by introducing EADAM. For the precise formulation of the mechanism, we refer the reader to Kesten (2010). EADAM is a subtle and complicated mechanism, but Kesten proves that 1) a student is never harmed by consenting to having her priority waived; 2) EADAM Pareto dominates DA; and 3) if all students consent, then EADAM is Pareto-efficient.[11]

In this section, we prove that the assignment made by EADAM when all students consent is a legal assignment. Since the EADAM assignment is Pareto efficient and a legal set of assignments is a lattice, this proves that EADAM Pareto dominates any other legal assignment. This characterization is analogous to Gale and Shapley's (1962) characterization of DA: the DA assignment is each man's favorite stable assignment. Our result demonstrates that the EADAM assignment is each student's favorite legal assignment.[12]

We will use the simplified EADAM mechanism (hereafter sEADA) introduced by Tang and Yu (2014). The key part of sEADA is the concept of an underdemanded school. For a given assignment μ, a school a is *underdemanded* if for every student i, $\mu_i R_i a$. There are several facts about underdemanded schools that critical for Tang and Yu's mechanism. First, under the DA assignment, there is always an underdemanded school. For example, the last school that any student applies to is an underdemanded school. Second, a student assigned by DA to an underdemanded school cannot be part of a Pareto improvement. Using these facts, Tang and Yu define sEADA iteratively as follows:[13]

The simplified Efficiency Adjusted Deferred Acceptance Mechanism (sEADA)

[11]Two recent papers have established some of the properties of EADAM. Dur and Morrill (2016) demonstrates that EADAM only Pareto improves DA when students submit the same preferences. When students are strategic, they provide an example in which at least one student is made worse off relative to DA in every Nash equilibrium. A corollary of this result is that, in equilibrium, a student may be harmed by consenting. Dur, Gitmez, and Yilmaz (2015) provides the first characterization of EADAM. They prove that it is the unique mechanism that is partially fair, constrained efficient, and gives each student the incentive to consent.

[12]For one-to-one matching problems, this characterization in terms of vNM stability was already known. Bando (2014) proves that EADA produces the male-optimal match in the unique vNM stable set for a marriage problem. His characterization does not directly apply to our problem as he restricts attention to one-to-one matching problems our paper studies many-to-one assignment problems. Prior to our work, it was thought that there was no connection between vNM stable sets in one-to-one matching problems and vNM stable sets in many-to-one problems is a college admissions problem (Ehlers, 2005).

[13]To be precise, this is the definition of sEADA when all students consent to allowing their priority to be violated.

Round 0: Run DA on the full population. For each student i assigned to an underdemanded school a, assign i to a; remove i; and reduce a's capacity by one.

Round k: Run DA on the remaining population. For each student i assigned to an underdemanded school a, assign i to a; remove i; and reduce a's capacity by one.

Tang and Yu (2014) prove that sEADA and Kesten's EADAM make the same assignment. We will prove that the sEADA assignment is a legal assignment and therefore Pareto dominates any other legal assignment. The following example provides the intuition for this result.

Example 2.

R_i	R_j	R_k	R_l	$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
b	a	a	b	i	j	k	l
a	c	c	d	k	l		
	b			j	i		

In Round 0 of sEADA, the DA assignment is:

$$\begin{pmatrix} i & j & k & l \\ a & b & c & d \end{pmatrix}$$

Note that i envies j's assignment. j envies i and k's assignment, and k envies i's assignment. However, no student strictly prefers d to her assignment. Therefore, d is an underdemanded school. sEADA assigns l to d and removes both the student and the school. Now the assignment problem is:

R_i	R_j	R_k	$\succ_a$	$\succ_b$	$\succ_c$
b	a	a	i	j	k
a	c	c	k	i	
	b		j		

The DA assignment of this problem (and therefore the Round 1 sEADA assignment) is:

$$\begin{pmatrix} i & j & k \\ b & c & a \end{pmatrix}$$

Student l has justified envy of the Round 1 sEADA assignment. Specifically, bP_ld and $l \succ_b i$. However, l cannot be part of a Pareto improvement of the Round 0 assignment since she is assigned to an underdemanded school. The consequence is that in any assignment where l is assigned to b, a student is made worse off relative to the Round 0 assignment. One can verify that this student blocks the new assignment with the DA assignment. Therefore, l has justified envy of b, but there is no legal assignment where she receives b. Therefore, the Round 1 assignment is legal as l's "harm" is not "redressable".

In the Round 1 assignment, the underdemanded schools are b and c. After removing i and j, all that remains is k and a; therefore, the assignment problem is trivial. The Round 2 sEADA assignment is to assign k to a. Therefore, the final sEADA assignment is:[14]

$$\begin{pmatrix} i & j & k & l \\ b & c & a & d \end{pmatrix}$$

A key point is that a student assigned to an underdemanded school cannot be part of a Pareto improvement.

Fact: (Lemma 1, Tang and Yu 2014) At the DA matching, no student matched with an underdemanded school is Pareto improvable.

This implies that if DA assigns i to an underdemanded school and i prefers school a to her DA assignment, then there is no legal assignment where she is assigned to school a.

Lemma 9. *Let $\mu = DA(P)$ and suppose μ_i is underdemanded. Then for any individually rational and nonwasteful assignment ν such that $\nu_i P_i \mu_i$, ν is blocked by μ.*

Proof. The key point is that an underdemanded students cannot be part of a Pareto improvement to μ. Let ν be any assignment such that $\nu_i P_i \mu_i$. ν does not block μ since μ has not justified envy. Suppose for contradiction that μ does not block ν. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments, and consider i's $(\overline{\mu}, \overline{\nu})$-cycle. By the General Decomposition Lemma, since $\nu_i P_i \mu_i$, $\nu_j P_j \mu_j$ for every

[14] Note that this example also demonstrates that the set of α-equitable assignments, as defined by Alcalde and Romero (2015), is not a legal set of assignments. There are two Pareto improvements of the DA assignment:

$$\mu = \begin{pmatrix} i & j & k & l \\ b & a & c & d \end{pmatrix} \qquad \nu = \begin{pmatrix} i & j & k & l \\ b & c & a & d \end{pmatrix}$$

Both μ and ν are α-equitable since they both Pareto improve the DA assignment. ν blocks μ (k prefers ν and has higher priority at a than does j); therefore, the set of α-equitable assignments is not a legal set of assignments.

student j in the cycle. Therefore, reassigning the students in the cycle to their assignment under ν is a Pareto improvement. However, this is a contradiction since an underdemanded student cannot be part of a Pareto improvement of μ. $\square$

Theorem 5. *The EADA assignment is a legal assignment.*

Proof. An equivalent way of defining sEADA is as follows. Set $P^1 = P$, and let μ^1 be $DA(P)$. Define U^1 to be the underdemanded students under assignment μ^1. We define preferences P^2 as follows. If $i \in U^1$, then move μ_i^1 to the top of i's preference list. If $i \notin U^1$, then leave i's preferences unchanged. In general, given P^k, we define $\mu_i^k = DA(P^k)$. We define U^k to be the underdemanded students under μ^k, and we modify P^k to create P^{k+1} as follows: if $i \in U^k$, then move μ_i^k to the top of i's preferences; otherwise, leave i's preferences unchanged. Note that $U^k \subseteq U^{k+1}$ since if a student does not envy school a under her true preferences, then the student does not envy school a when we move her assignment to the top of her preference list. The process stops once all students are underdemanded. It is straightforward to verify that this is equivalent to the sEADA procedure.

We prove by induction that for each integer k, (a) $\mu^k \in S^k$ and (b) if $i \in U^k \setminus U^{k-1}$[15] and ν is an assignment such that $\nu_i P_i \mu_i^k$, then $\nu \notin B^k$. μ^1 is the DA assignment which has no justified envy. Therefore $\mu^1 \in S^1$. Lemma 9 establishes part (b) of the base step.

Now suppose $k > 1$. Note that if $U^k = U^{k-1}$ then $\mu^k = \mu^{k-1}$ and the result holds trivially. Now suppose $U^k \setminus U^{k-1} \neq \emptyset$. First, we show that $\mu^k \in S^k$. If i has justified envy of μ^k under preferences P, then $P^k \neq P$ (since μ^k has no justified envy under preferences P^k). Therefore, by construction $i \in U^{k-1}$. However, if $i \in U^{k-1}$ and ν is an assignment such that $\nu_i P_i \mu_i^k$, then by the inductive hypothesis, $\nu \notin B^{k-1}$. Therefore, $\mu^k \in \pi(B^{k-1}) = S^k$. Now suppose $i \in U^k \setminus U^{k-1}$. Since i is a "new" underdemanded student, we have not yet modified her preferences (we will do so in the next round). In particular, $P^k = P$. Therefore, Lemma 9 applies. Lemma 9 says that if i is an underdemanded student, and ν is an assignment that i prefers to $\mu^k = DA(P^k)$ (relative to preferences P^k; specifically if $\nu_i P_i^k \mu_i^k$), then the DA assignment blocks ν (using preferences P^k). To complete our proof, we must show that if ν is an assignment such that $\nu_i P_i \mu_i^k$, then $\nu \notin B^k$. Since $P_i = P_i^k$, $\nu_i P_i^k \mu_i^k$. But by Lemma 9 since i is assigned to an underdemanded school (under preferences P^k) by μ^k, if ν is an assignment such that $\nu_i P_i^k \mu_i^k$, then by Lemma 9, ν is blocked by μ^k. Since $\mu^k \in S^k$, $\nu \notin \pi(S^k) = B^k$.

[15] We set $U^0 = \emptyset$.

Since each $S^k \subset S^n$, this proves that the sEADA assignment is in S^n and therefore is legal. $\square$

6 Petty Envy

Legality is a pragmatic consideration. The market designer likely views this as a constraint rather than an objective. In school assignment, the objectives the designer typically balances are fairness and efficiency. The literature calls an assignment unfair if a student i has justified envy of a student j at some school a.[16] But what if it is impossible to assign i to a but it is possible to assign j to a? Ignoring for the moment why it might be possible for one student and not for another, if we honor i's objection, it does not help i but only harms j. Is it fair to not assign j to a just to keep i from being jealous? Typically, people would not call this fair and instead would call it petty.

The key point is that we cannot determine if an assignment is fair in isolation; instead, we must ask if a *set* of possible assignments is fair. Given a set of assignments A, we define an assignment to be **A-fair** if it is not blocked by any assignment in A. Using our notation from legality, we denote the set of A-fair assignments by $\pi(A)$. We define A to be a fair set of assignments if it satisfies the following three conditions. First, no assignment in A is blocked by another assignment in A (each assignment in A is A-fair). Second, no assignment in A is blocked by an A-fair assignment. Third, we require that if an assignment is not blocked by any A-fair assignment, then it is in A. We interpret the A-fair assignments as "possible" and all other assignments as "impossible". The first condition says that a fair assignment should be possible. The second says that a fair assignment should not be blocked by any possible assignment. The third condition says that if an assignment is only blocked by impossible assignments, then it is fair. To not include such an assignment would be petty.

Definition 5. F is a **fair set of assignments** if:

1. Each assignment in F is F-fair: $F \subseteq \pi(F)$.

2. No assignment in F is blocked by an F-fair assignment: $F \subseteq \pi(\pi(F))$.[17]

[16] Justified envy was introduced by Balinski and Sonmez (1999) as the definition of a fair assignment. Following Abdulkadiroglu and Sonmez (2003), the literature often uses fairness and justified envy interchangeably.

[17] At first glance, some people mistakenly think that condition (2) implies condition (1). It does not. For example, consider the classic roommates problem presented in Gale and Shapley (1962) as an example of an assignment problem

3. If μ is not blocked by any F-fair assignment, then $\mu \in F$: $\pi(\pi(F)) \subseteq F$.

If condition 3 is violated, we will say the set of assignments is **petty**.

Mathematically, conditions (2) and (3) imply that if F is fair, then $\pi(\pi(F)) = F$. We present them as separate conditions so that it is easier to interpret. Our next theorem shows that our concepts of legality and fairness are equivalent.

Theorem 6. *The set of legal assignments is the unique fair set of assignments.*

Proof. Let F be a fair set of assignments, and for each integer $k \geq 1$, let S^k and B^k be defined by Equations (2) and (3) on Page 7. Applying the Monotonicity Fact twice, for any set of assignments A and B, if $A \subseteq B$, then

$$\pi(\pi(A)) \subseteq \pi(\pi(B)) \tag{5}$$

We prove by induction that $S^k \subseteq F$ for any $k \geq 1$. By the Justified Envy Fact, $S^1 \subseteq \pi(\pi(F))$. By the definition of fairness, $\pi(\pi(F)) = F$. Therefore, $S^1 \subseteq F$. Suppose $S^{k-1} \subseteq F$. By Equation 5, $\pi(\pi(S^{k-1})) \subseteq \pi(\pi(F))$. By construction, $\pi(\pi(S^{k-1})) = S^k$. Since, $\pi(\pi(F)) = F$, $S^k \subseteq F$. Therefore, for every $k \geq 1$, $S^k \subset F$. Therefore, $\pi(F) \subseteq \pi(S^k) = B^k$ (by the Monotonicity Fact). Since F is fair, $F \subseteq \pi(F)$. Therefore, $F \subseteq B^k$. We conclude that for every $k \geq 1$

$$S^k \subseteq F \subseteq B^k$$

Since $S^n = B^n$ by Theorem 2, we conclude that $F = S^n$ and therefore there is a unique fair set of assignments. □

There is a natural connection between our notion of fairness and von Neumann Morgenstern stable sets (hereafter vNM stable). A set of assignments S is **vNM stable** if $S \subseteq \pi(S)$ (internal stability) and $\pi(S) \subseteq S$ (external stability). Equivalently, S is vNM stable if and only if $S = \pi(S)$. It is

with no stable solution. Let F is the set of all possible roommate assignments. Since every assignment is blocked by some assignment, $\pi(F) = \emptyset$. However, trivially $\pi(\emptyset) = F$. Therefore, $F \subseteq \pi(\pi(F))$ but $F \not\subseteq \pi(F)$. Of more direct interest to this paper, prior to Theorem 2, it was hypothetically possible that $S^n \neq B^n$. If this were the case, then $\pi(B^n) = S^n$. Therefore, $B^n \not\subseteq S^n$. But $\pi(\pi(B^n)) = B^n$. Therefore, for set B^n, condition (2) is satisfied but not condition (1).

straightforward to verify that if S is vNM stable, then S is fair. However, in general vNM is a stronger condition than fairness. In particular, even when the core is non-empty, a vNM stable set need not exist (see, for example, Lucas 1968). However, whenever the core is non-empty, a fair set exists and is a superset of the core. However, for the school assignment problem, the two conditions are equivalent.

Theorem 7. *A set of school assignments is fair if and only if it is vNM stable.*

Proof. vNM stability is a stronger condition than fairness. Suppose that S is a fair set of assignments. By the proof of Theorem 6, $S = S^n$. By Theorem 2, $S^n = B^n = \pi(S^n)$. So indeed, $\pi(S) = S$, and S is stable. □

An immediate corollary is that for the school assignment problem there exists a unique vNM stable set of assignments.

Corollary 8. *There exists a unique vNM stable set of assignments.*

After completing our proof of Theorem 2, we discovered the connection between legality, pettiness, and vNM stability. Interestingly, several papers have consider vNM stability in regards to the marriage problem[18] and have found similar results. Ehlers (2007) characterizes vNM stability for the marriage problem by showing that any vNM stable set V is a maximal set satisfying the following properties: (a) the core is a subset of V, (b) V is a distributive lattice, and (c) the set of unmatched students is the same for all matchings belonging to V. Ehlers remarks that his characterization does not extend to the many-to-one matching problem and that there need not be any relationship between vNM stable sets of a many-to-one problem and vNM stable sets of its corresponding one-to-one matching problem.[19] Motivated by Ehlers (2007), Wako (2010) introduces an algorithm that

[18] A marriage problem is a one-to-one matching problem. In our framework, it is equivalent to each school having a capacity of one.

[19] Our results show that there is a unique vNM stable set in any many-to-one matching problem and that it has the same mathematical structure as Ehler's characterization. We do not view this as a contradiction of Ehler's statement. From the working paper Ehlers (2005), it is clear that he means that there is no direct relationship between the vNM stable set of a many-to-one matching problem and the vNM stable set of the "clone" economy made by creating q_a clones of each school a and modifying preferences in a specific way. The cloning procedure is useful for many problems, but in this instance it creates artificial structure that alters the set of vNM stable assignments.

proves that there exists a unique vNM stable set for any marriage game.[20] Bando (2014) proves that in the marriage problem, EADA produces the male-optimal match in the unique vNM stable set.

Kloosterman and Troyan (2016) recently introduced a new fairness concept called essentially stable. We refer the reader to their paper for a formal definition of essentially stable, but intuitively, an assignment is essentially stable if whenever a student i has justified envy of school a, placing i at a and removing a student leads to a succession of appeals that ultimately leads to the removal of student i from school a. Kloosterman and Troyan (2016) and the current paper were developed independently. Essentially stable and petty envy are similar in spirit, Kloosterman and Troyan (2015) demonstrate that not all legal assignments are essentially stable.

7 Conclusion

When a school board chooses an assignment mechanism, it typically balances strategyproofness, efficiency, and fairness. However, a critical pragmatic consideration for any board is which of the possible assignments are legal. We show that there is a unique set of legal assignments, and that there is always a unique Pareto efficient assignment that is legal. Prior to our work, it was thought that there was no "silver bullet" solution to the school assignment problem as it is impossible for a mechanism to be both efficient and eliminate justified envy (Abdulkadiroglu and Sonmez, 2003). However, we show that the only envy of a legal assignment is either unjustified or else is petty. Therefore, the set of legal assignments satisfy a natural interpretation of fairness. Combined, these results suggest that the assignment made by Kesten's EADA is the ideal school assignment. It is the unique assignment that is legal and Pareto efficient. It is fair in a meaningful way, and it Pareto dominates any other fair or legal assignment.

[20] Wako actually showed existence and uniqueness in an earlier note, Wako (2008). Our results were all discovered independently of Wako's work, but the approach he takes in Wako (2008) is similar to the mathematical approach we have taken in this paper. His work does not directly apply to our paper as it is restricted to one-to-one matches.

8 Appendix

Here we prove Theorem 2: at the conclusion of the iterative process every assignment has been determined to be legal or illegal. As a reminder, for any set of assignments A, $\pi(A)$ are the assignments not blocked by any assignment in A. We defined S^1 to be the set of assignments with no justified envy. In general, $B^k = \pi(S^k)$ and $S^{k+1} = \pi(B^k)$. We showed that $S^1 \subseteq \ldots \subseteq S^n \subseteq B^n \subseteq \ldots \subseteq B^1$. We defined n as any integer such that $S^n = S^{n+1}$. We also proved that S^n is a Lattice and that there exists a Pareto efficient assignment in S^n (specifically Kesten's EADA assignment).

Theorem 2: $S^n = B^n$.

Proof. Suppose for contradiction that there exists an assignment $\nu \in B^n \backslash S^n$. Since $\nu \notin S^{n+1} = S^n$, ν is blocked by some student i with assignment $\mu \in B^n$. Let $a = \mu_i$. Note that there does not exist an assignment $\phi \in S^n$ such that $\phi_i = a$. Otherwise, i would block ν with ϕ in which case $\nu \notin B^n$.

In the proof of Theorem 5, we demonstrated the following: for any student j, if b is j's assignment under EADA and c is a school such that cP_jb, then there does not exist an assignment $\lambda \in B^n$ such that $\lambda_j = c$. Since $\phi_i = a$ and $\phi \in B^n$, it follows that i strictly prefers her EADA assignment to a (a is not i's EADA assignment since the EADA assignment is in S^n). We can define the school proposing EADA assignment analogously.[21] The same logic yields that i strictly prefers a to her assignment under the school proposing EADA as any assignment in which i receives a worse assignment than under the school proposing EADA is blocked by the school-proposing EADA assignment. In particular, both $\{\phi \in S^n | \phi_i P_i a\}$ and $\{\phi \in S^n | a P_i \phi_i\}$ are non-empty. Since S^n is a lattice, the following are well defined.

$$\overline{\mu} := \min_{>} \{\phi \in S^n | \phi_i P_i a\} \tag{6}$$

and

$$\underline{\mu} := \max_{>} \{\phi \in S^n | a P_i \phi_i\} \tag{7}$$

Claim 1: For any $\phi \in S^n$, either $\phi R \overline{\mu}$ or else $\underline{\mu} R \phi$.

[21] Specifically, create q_a clones of each school a. Each clone has the same priorities over students as does a. Each student prefers lower numbered clones to higher numbered clones. Run sEADA with the clones proposing to students.

This is an immediate consequence of the fact that $\phi_i \neq a$ for any $\phi \in S^n$. Therefore, either $\phi_i P_i a$, in which case $\phi R \overline{\mu}$ or else $a P_i \phi_i$ in which case $\underline{\mu} R \phi$.

Let $k = \min_{\succ_a} \overline{\mu}_a$ (the lowest priority student assigned to a by $\overline{\mu}$). Consider the following generalization of the Deferred Acceptance algorithm. For each student l, define all schools that l strictly prefers to $\overline{\mu}_l$ to have rejected l. Next, we reject k from a. This begins what we will refer to as a vacancy chain. We only allow a student l to apply to a school b if there exists a $\phi \in B^n$ such that $\phi_l = b$ (we call such a school **achievable** for the student). We have k apply to her favorite achievable school that has not yet rejected her. Each time a school receives a new application, if it has an available seat, then the school does not reject a student. If it does not have an available seat, then it rejects the lowest ranked student. Each time a student is rejected, it applies to her next favorite achievable school. The process ends when there is no new rejection. Let ϕ be the assignment that results from this process.

Claim 2: $\phi R \underline{\mu}$.

If not, then let l be the first student in the vacancy chain rejected by her assignment under $\underline{\mu}$. l is only rejected after some student m applies to $\underline{\mu}_l$. m has not been rejected by $\underline{\mu}_m$ since l is the first student rejected by her assignment under $\underline{\mu}$. Since m applied to $\underline{\mu}_l$ before applying to $\underline{\mu}_m$, by revealed preference $\underline{\mu}_l P_m \underline{\mu}_m$. Since $\underline{\mu}_l$ rejected l in favor of m, $m \succ_{\underline{\mu}_l} l$. Since m can only apply to an achievable school, there exists a $\phi \in B^n$ such that $\phi_m = \underline{\mu}_l$. Therefore, m blocks $\underline{\mu}$ with ϕ, contradicting the fact that $\underline{\mu} \in S^{n+1}$ and therefore is not blocked by any assignment in B^n.

Claim 3: the vacancy chain concludes when the first time a student applies to a.

There are only three possible ways for the vacancy chain to end: 1) a student applies to a, 2) a student applies to a school b and $|\overline{\mu}_b| < q_b$, and 3) a student is rejected by all of her achievable schools. If the third case occurred, then there would exist a student l such that $\overline{\mu}_l \neq \emptyset$, but $\phi_l = \emptyset$. Since $\overline{\mu}_l \neq \emptyset$, $\underline{\mu}_l \neq \emptyset$ by the Rural Hospital Theorem. Since $\phi_l \neq \underline{\mu}_l$ and $\phi_l R_l \underline{\mu}_l$, $\phi_l P_l \underline{\mu}_l$. But this contradicts the individual rationality of $\underline{\mu}$ since $\phi_l = \emptyset$. Similarly, if the second case occurred, then for some student l, $\phi_l = b$ and $|\overline{\mu}_b| < q_b$. By the Strong Rural Hospital Theorem, $\overline{\mu}_b = \underline{\mu}_b$. Since $\phi_l R_l \underline{\mu}_l$ and $b \neq \underline{\mu}_l$, $b P_l \underline{\mu}_l$. But this implies that $\underline{\mu}$ is wasteful as b has available seats under $\underline{\mu}$ which is a contradiction. Therefore, only the first case is possible. The vacancy chain concludes the first time a student applies to a.

Let l be the student who ends the vacancy chain. Note that $\phi_a = \overline{\mu}_a \cup \{l\} \setminus \{k\}$

Claim 4: $\phi \in S^n$.

Suppose for contradiction that i blocks ϕ with $\nu \in B^n$. Let $\nu_i = b$. We first show that $\phi_b \geq_b \overline{\mu}_b$. This is clear for $b \neq a$ as the rankings of such a school can only improve when it receives a new application. For school a, we must show that $l \succ_a k$. a is achievable for l, so there exists a $\sigma \in B^n$ such that $\sigma_l = a$. Since $\overline{\mu} \in S^n$ and $\sigma \in B^n$, μ and σ do not block each other.[22] Since $\overline{\mu}_l P_l a = \sigma_l$, by the General Decomposition Lemma, $\sigma_a >_a \overline{\mu}_a$. Therefore, for any student $p \in \overline{\mu}_a \setminus \sigma_a$ (and by the Rural Hospital Theorem such a student exists), $l \succ_a p$. However, k was chosen to be the lowest priority student at a. Therefore, $p \succeq_a k$, and consequently $l \succ_a k$. So indeed, $\phi_b \geq_b \overline{\mu}_b$. Since i blocks ϕ with ν, i is ranked higher than some student in ϕ_b. Since $\phi_b \geq_b \overline{\mu}_b$, i must be ranked higher than some student at $\overline{\mu}_b$. Therefore, if $\phi_i R_i \overline{\mu}_i$, then i would block $\overline{\mu}$ with ν. Since ν does not block $\overline{\mu}$, it must be that $\overline{\mu}_i P_i \phi_i$. But this means that i was rejected by b during the vacancy chain and therefore every student at ϕ_b has higher priority than does i. This contradicts the assumption that i blocked *phi* with ν which requires that i has higher priority than at least one student at $\nu_i = b$.

Claim 5: $\phi = \underline{\mu}$

$\overline{\mu}_k = a P_k \phi_k$. Therefore, it cannot be that $\phi R \overline{\mu}$. Since $\phi \in S^n$, it follows from Claim 1 that $\underline{\mu} R \phi$. Since, $\phi R \underline{\mu}$ by Claim 2, we conclude that $\phi = \underline{\mu}$.

However, this is a contradiction. The vacancy chain concludes the first time a student applies to a. For student i, $\overline{\mu}_i P_i a P_i \underline{\mu} = \phi$. Therefore, i participated in the vacancy chain. a is achievable for i and i strictly prefers a to ϕ_i. Therefore, i must have applied to a at some point in the vacancy chain and must have been rejected by a. However, this is a contradiction as the vacancy chain stops the first time a student applies to a. □

References

Abdulkadiroğlu, A., P. A. Pathak, A. E. Roth, and T. Sönmez (2005): "The Boston public school match," *American Economic Review*, pp. 368–371.

[22] Moreover, it is straightforward to verify that they are also nonwasteful and individually rational.

Abdulkadiroğlu, A., and T. Sönmez (2003): "School choice: A mechanism design approach," *The American Economic Review*, 93(3), 729–747.

Alcalde, J., and A. Romero-Medina (2015): "Strategy-Proof Fair Schoo Placement," *mimeo.*

Balinski, M., and T. Sönmez (1999): "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, 84(1), 73–94.

Bando, K. (2014): "On the existence of a strictly strong Nash equilibrium under the student-optimal deferred acceptance algorithm," *Games and Economic Behavior*, 87, 269 – 287.

Cantala, D., and S. Papai (2014): ""Reasonably and Securely Stable Matching"," *"mimeo"*.

Dur, U., A. Gitmiz, and O. Yilmaz (2015): "School Choice Under Partial Fairness," *mimeo.*

Ehlers, L. (2007): "Von Neumann-Morgenstern stable sets in matching problems," *Journal of Economic Theory*, 134(1), 537 – 547.

Gale, D., and L. Shapley (1962): "College admissions and the stability of marriage," *American Mathematical Monthly*, 69(1), 9–15.

Hessick, F. A. (2007): "Standing, Injury in Facts, and Private Rights," *Cornell L. Rev.*, 93, 275.

Kesten, O. (2010): "School choice with consent," *The Quarterly Journal of Economics*, 125(3), 1297–1348.

Kloosterman, A., and P. Troyan (2016): "Efficient and Essentially Stable Assignments," *mimeo.*

Lucas, W. (1968): "A Game With no Solution," *Bulletin of the American Mathematical Society*, 74, 237 – 239.

Roth, A. (1982): "The economics of matching: Stability and incentives," *Mathematics of Operations Research*, pp. 617–628.

Roth, A., and M. Sotomayor (1992): *Two-sided matching: A study in game-theoretic modeling and analysis.* Cambridge Univ Pr.

Sönmez, T., and U. Ünver (2010): "House Allocation with Existing Tenants: A Characterization," *Games and Economic Behavior*, 69(2), 425–445.

Tang, Q., and J. Yu (2014): "A new perspective on Kesten's school choice with consent idea," *Journal of Economic Theory*, 154, 543–561.

von Neumann, J., and O. Morgenstern (1953): "Theory of Games and Economic Behavior," *Princeton University Press*.

Wako, J. (2008): ""A Note on Existence and Uniqueness of vNM Stable Sets in Marriage Games"," *Match-Up Workshop Proceedings of ICALP 2008*, pp. 157–168.

——— (2010): "A Polynomial-Time Algorithm to Find von Neumann-Morgenstern Stable Matchings in Marriage Games," *Algorithmica*, 58(1), 188–220.

Zhou, L. (1994): "A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation," *Games and Economic Behavior*, pp. 512–526.