Game Theory

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Chapter 1

Introduction

When I was an undergraduate, I wanted to take game theory because I thought we would spend the semester talking about how to play a variety of board games. This would be a great class, but sadly, I've never heard of a class like this. Game theory doesn't spend any time talking about what normal people call games. So if we are not going to talk about board games, what do we mean by a game?

When you start learning economics, none of the agents interact. Prices are fixed and a person must decide what to buy. A monopolist doesn't interact with any other firms because there are no other firms. I will call something a **decision** if the outcome only depends on your preferences. What ice cream should you order? That's a decision. Which stock should you buy? That's a decision. These can be complicated and involve uncertainty, but a decision always has a right answer.

A **game** is when people (or firms) interact. The key is that the outcome depends on the choices made by both people. Should you ask a person out on a date? That's a game. It depends on how they feel about you and how they will respond. This is actually a good way of telling if something is a game.

> A game is any decision where the answer is *it depends*.

Should I take my dog for a walk now or later? This is a decision. It doesn't depend on anything except the costs and benefits for me. Should I work hard on a group project? Well, that depends. If no one else is going to do the work and I care about my grade, then yes, I should. If other people are going to do all the work and I think they will do a good job, then no, I shouldn't. Since my choice depends on what I believe their choices will be, and their choices depend on my choice, this is a game.

In games, typically, there is no right or wrong answer. For example, you might be tempted to ask, if United Airlines lowers their prices, how will this change its revenue? This question makes sense in introductory economics: we think lowering prices will increase the number of tickets it sells and we look at the elasticity of demand to see if revenue will increase or decrease. By the end of this course, I hope you recognize that the question itself doesn't make sense. It has no answer. Why? Well, if United Airlines lowers its prices, then Delta might respond. Delta might lower their prices. Delta might even raise their prices! Why? If the customers who are price sensitive buy a ticket from United, that means that the remaining customers don't care very much about price (perhaps their ticket is paid for by their company), and it might be a good idea for Delta to raise prices! The point is that we cannot answer this question by only looking at United. If United airlines lowers their prices, how will it change its revenue? It depends. It depends on how Delta will respond. There is not a "right" answer to this question. We will not be looking for "right" answers; instead, we will be looking for what are potential outcomes of this interaction.

You can see that what we are going to call a game is different than what normal people call a game. Normal people call Blackjack a game (when I am not teaching game theory, I do to). If you haven't played before, you are dealt two cards and you get to see one of the dealer's cards. You can hit, get another card, or stay. The rub is that if you go over 21 you bust and automatically lose. If you stay, then the dealer flips over her cards and it's her decision. Except the dealer has no choice in the matter. Each casino has rules the dealer must follow. For example, in many casinos, the dealer must hit until she either reaches 17 or she busts. Since the dealer has no control over her decision, when you are making your decision, there is always a right thing to do. It may be hard to calculate (should I hit if I have 15 and the dealer shows a 2?), but there is always a right answer as to which decision will yield the highest expected payoff.

Compare this to poker. In poker, there are no casino rules about how a player must play. Each player is free to bet, call, or fold as she sees fit. When a person bets into you, should you call or fold? There is no right or wrong answer to this question. If the person is a "loose" player and bets a lot, and you have a pretty good hand, then you should probably call. If the person is a "tight" player and never bets unless she has a winning hand, then you should probably fold. But in all situations, there is no right or wrong thing to do. It depends on the other player's strategy.

Why is this economics? In introductory and intermediate economics classes, we normally look at perfectly competitive markets or monopolies. In a perfectly competitive market, there are so many other firms that no one firm can affect the price. A firm doesn't have to decide what price to set. There is a market price and they decide how much to sell at that price. How much do I sell at that price? My choice doesn't affect any other firm and no other firm's choice affects the price I get, so it is simply a decision based on my opportunity cost of production. A monopolist does choose a price. But a monopolist, by definition, has no competition. So all a monopolist needs to consider is the demand for their product and the cost of production. It doesn't have to worry about how its competitors will react. It has not competitors. These decisions may be hard, but they are decisions; they don't depend on what other firms will do.

But a little secret about economics is there aren't many competitive markets and there are perhaps no monopolists. When I think about all the things I buy during a month, I can't think of a single product where the market is either perfectly competitive or a monopoly. Maybe gas is competitive. But is Apple a monopolist over the iPhone? In one sense, yes. They are the only firm that can sell iPhones. But they really aren't a monopoly. What price they charge is a game. There is no right or wrong answer. It depends on how Samsung will respond pricing the Galaxy. Will I buy an iPhone if they charge me $800? Not if Samsung is charging $400 for a Galaxy, but yes I will if Samsung decides to charge $1,200. How much should United charge for a flight from Raleigh to Chicago? Well, it depends on what Delta is charging for the same route. Should United lower its price by $50? It depends on how Delta will react. If Delta won't change their price, then this might be a good idea. But if Delta will immediately drop its price by the same amount, then there is probably little benefit but a clear cost. Delta might even raise its prices![1] The answer depends.

This is the purpose of this class. There are many economic situations (and life situations!) where the answer to what you should do is "it depends". We want to figure out a way beyond that. When I am making my choices based on what I think you will do, and you are making your choices based on what I think you will do, then we want to figure out a way to predict what will happen.

[1] Why? If the customers who are price sensitive buy a ticket from United, that means that the remaining customers don't care very much about price (perhaps their ticket is paid for by their company), and it might be a good idea for Delta to raise prices!

Chapter 2

Nash Equilibrium

The challenge of game theory is what happens when you get stuck in a logical loop. In rock, paper, scissors, if you think I'm going to play rock, then you'll play paper. But if I think you'll play paper, then I won't play rock. But if you think I won't play rock, then you won't play paper. But if I think you won't play paper, then I'll play rock. At this point, both your head and mine start to hurt. Basically, our goal in this course is to figure out what might happen when we get stuck in one of these loops.

This course doesn't have many definitions - that's the good news - but the ones it has tend to be hard. Nash Equilibrium is basically the only definition we will use in this course. It's a definition - true - but it's really a new way of thinking about a problem. I prefer to think of it as a thought process. Like inductive logic or other thought processes, you have to train yourself to think this way. For most people, a Nash Equilibrium is confusing. It should be. It's a new approach to thinking, and it's subtle. That means it's going to be confusing at first, and it's going to take some practice and deliberate effort so that thinking this way becomes natural.

2.1 Equilibrium Logic

First, what do we mean by *equilibrium*? We are trying to predict what might happen. We're looking for something where we think "ok, it might happen this way over, and over, and over again." It's helpful to keep in mind that there are two different ideas - an equilibrium versus not an equilibrium. We define what an equilibrium is, but in general, we usually feel better when we figure out that something is **not** an equilibrium. Not an equilibrium means someone is making a mistake. If we play the game over and over and over again, eventually the person making the mistake will figure it out, so we are generally more confident when we figure out that something is not an equilibrium than we are when we figure out it is. For example, suppose Jerry is supposed to always play rock and Coco is supposed to always play paper. Is this an equilibrium? Jerry is making a mistake. If they play this way over and over and over again, Jerry is going to get sick of losing. Eventually he will figure out "hey, I could win if I just play scissors." I am not sure what the equilibrium is, but I am sure that if you and I play rock paper scissors a bunch of times, what **won't** happen is you always play rock and I always play paper.

Here is the thought process that John Nash gave us which is our way out of the logical loop. He asked: "What if I knew what you were going to play? Would I change what I am going to play?" If the answer to the second question is "yes", then we don't think this will be the regular outcome of the game. This is not an equilibrium. But what if both people answer no? In otherwords, suppose you tell me what you are going to play, and I don't want to change what I choose. And I tell you what I'm going to play, and you don't want to change your choice. When both of these things happen at the same time, no one wants to change. Now, it is possible that we will do the same thing over and over again. This is the idea behind a **Nash Equilibrium**

2.2 Strategy

This is one of the most essential concepts in game theory. It is a little more subtle than you might initially think, mainly because strategy is a word we all use regularly and are comfortable. In game theory, we will use strategy in a very specific way.

> Definition
>
> A player's **strategy** is her plan.

Okay, that might not be super helpful right now, but it is a good way to think about it as we go along. Right now we are going to start with simple games where each player is choosing an action. So, for now, it is fine if you think of a strategy as what action a player chooses. But be aware that pretty soon strategies will be more complicated than this.

2.3 Best Response

In general we don't know what another player's strategy is. But we can ask ourselves a reasonable thought question: What if we did? What would we choose? The best choice, meaning the one that gives us the highest payoff, is what we call our best response.

Definition

For player A, strategy s^* is a **best response** to strategy s' if playing s^* gives player A the highest possible payoff whenever the other player plays strategy s'.

For a strategy s and a player i, I will typically use

$$BR_i(s)$$

to denote the set of best responses for player i to strategy s. This is a key point.

Key Fact

There can be more than one best response to a strategy.

The writing is a little awkward, but once you get comfortable with it, this is a simple idea. Suppose we are playing rock-paper-scissors. The best response to strategy "paper" would be to play strategy "scissors". Let me introduce a simple game. There is no story behind this game. It is only intended to help us understand what a best response is. Typically, I will describe a game by using a table as below.

		Elizabeth	
		H	T
Lou	H	2,-1	-1,1
	T	-1,1	1,-1

The first number is Lou's payoff. The second number is Elizabeth's. To make sure we're on the same page, what this table indicates is that if Lou plays H and Elizabeth plays H, Lou will get 2 and Elizabeth will get -1. If Lou plays H and Elizabeth plays T, then Lou gets -1 and Elizabeth gets 1. Below we have underlined Lou's best responses to Elizabeth's actions. For example, if Elizabeth plays T, then Lou can get -1 by playing H or 1 by playing T, so T is the best response to H.

Lou \ Elizabeth	H	T
H	$\underline{2}$,-1	-1,1
T	-1,1	$\underline{1}$,-1

One important point is that a best response does not need to be unique. Consider the game below Again, we have underlined player 1's best responses.

Lulu \ Elizabeth	C	D	E
A	$\underline{5}$,1	2,2	4,6
B	$\underline{5}$,0	$\underline{7}$,2	$\underline{6}$,0

Notice that if Elizabeth plays C, that either A or B give Lou a payoff of 5. Therefore, either choice is a best response.

2.4 Nash Equilibrium

We are now ready to introduce the most important definition in game theory.

> **Definition**
>
> A strategy s_1 for player 1 and a strategy s_2 for player 2 are a **Nash Equilibrium** if s_1 is a best response to s_2 and s_2 is a best response to s_1.

We typically just refer to this as a **mutual best response**. There are a couple of different (but equivalent) ways of thinking about a Nash equilibrium. These will help give us different approaches to solving games. One way of thinking about a Nash equilibrium is as follows.

In a typical game, you have to make your choice without knowing what the other person is doing. But what if after each player chooses her strategy, I tell you your opponent's strategy and offer you a do-over? In a Nash equilibrium, you wouldn't change your strategy. Why? Because you were already playing a best response, so no alternative could give you a higher payoff.

Definition

Strategies are a **Nash Equilibrium** if no player would take a do-over if they were offered one.

Another idea that will be helpful is what we call an incentive to deviate. A player has an **incentive to deviate** if a different choice would give her a higher payoff. Incentive to deviate is closely related to best response. Specifically, if a player has an incentive to deviate, then she cannot be playing a best response! Therefore, if a player does *not* have an incentive to deviate, then she is playing a best response. Therefore, we can define a Nash equilibrium in the following way:

Definition

Strategies are a **Nash Equilibrium** if no player has an incentive to deviate.

All three definitions are equivalent. This isn't always immediately obvious, so it is worth it to spend a little time thinking about this to make sure you agree.

2.5 Solving for Nash Equilibria

The reason we give two definitions - one in terms of best responses and one in terms of incentive to deviate - is that each definition suggests a different approach to solving for a Nash equilibrium. Under the "incentive to deviate" approach, we take a guess and then check to see if any player has an incentive to deviate. The "best response" approach is constructive. We look at each strategy; determine a best response; then find a mutual best response. If we are fortunate enough to be able to write a game in **normal form** (which just means the tables we have already seen), then we can always solve a game by the best response approach. Here are some classic games along with their solutions. In each, we have underlined the best response to each strategy.

Example 1 (Prisoner's Dilemma)**.** The Prisoner's Dilemma is the most famous game. Two people, Elizabeth and Lou, are arrested by the police. They are innocent, but the police puts each in a separate interrogation room and offers each person the following deal. If your friend confesses ("Fink") and you do not ("Quiet"), then you will go to prison for a very long time. If you both confess, then you will go to prison for a medium length of time. If neither of you confesses then you go to prison for a short length of time. However, if you confess and your friend does not, then the police will not send you to prison. We represent this decision in the normal form game below. Note that the numbers do not mean anything beyond each person prefers a big number to a small number.

		Elizabeth	
		Q	F
Lou	Q	3,3	0,$\underline{4}$
	F	$\underline{4}$,0	$\underline{1}$,$\underline{1}$

There are a couple of important things to notice about this example. First, there is a unique equilibrium, and in this equilibrium, both people confess. Remember, these people are both innocent! Second, note that this equilibrium is **inefficient** in the following sense. Both players would be strictly better off if they both remained quiet. Yet both remaining quiet is not an equilibrium as each player has the incentive to deviate.

The Prisoner's Dilemma is a really important game, so let me restate that. If both people stay quiet, then both are better off than when they both Fink. Staying quiet is unambiguously a better outcome for both players than when they both confess. But do we think we can sustain both staying quiet? It is a pretty good outcome, the Elizabeth is getting 3, but eventually I believe she will figure out that she can do even better if she Finks. Specifically, if Elizabeth believes that Lou will stay quiet, it is better for her to Fink than to stay Quiet. Therefore, I am skeptical that staying quiet can be sustained. Each person has an incentive to change what they are doing! However, if both are confessing, then neither person has a reason to change what they are doing. This is what we mean by an equilibrium.

The Prisoner's Dilemma is adversarial. Not all games are though. The next example is a game in which both players are trying to coordinate.

Example 2 (Battle of the Sexes)**.** In this game, Elizabeth and Lou are trying to meet up to go on a play date. The most important thing to each is that they get to spend time together. But, that being said, Elizabeth would prefer to go watch tennis (T) while Lou would prefer to go watch ballet (B). If they end up choosing different activities, then both will be so sad that they missed each other that they will get no enjoyment out of the activity. The normal form game is below.

		Elizabeth	
		B	T
Lou	B	$\underline{2},\underline{1}$	0,0
	T	0,0	$\underline{1},\underline{2}$

Notice that there does not have to be a unique equilibrium!

The next game is what we call a zero-some game. If one person wins, then the other person loses. The game is meant to be a simple version of rock-paper-scissors.

Example 3 (Matching Pennies)**.** In this game, Elizabeth and Lou must flip over a coin either heads (H) or tails (T). If they flip over the same thing, Lou wins. If not, then Elizabeth wins.

		Elizabeth	
		H	T
Lou	H	$\underline{1}$,-1	-1,$\underline{1}$
	T	-1,$\underline{1}$	$\underline{1}$,-1

Notice that it looks like there is no equilibrium. We will find soon that there is one, but we will need to define a new type of strategy first.

The next two games are important enough that we will take more time to study and think about them. They are old games that changed the way economists thought about competition among firms.

2.5.1 Cournot's model of duopoly

There are two firms, Firm 1 and 2. Each firm must simultaneously and independently determine the quantity of goods they wish to sell (Firm i chooses quantity q_i). The goods the firms make are identical, and the price they will receive is determined at the market level. The market quantity of goods is the total number of goods produced ($Q = q_1 + q_2$) and the market clearing price for the goods is given by the equation $P = A - Q$ where A is a constant. Each firm has the same constant marginal cost C.

Notice that we cannot solve this game by writing down a normal form game. The quantity can be any number, so there are an infinite number of choices! We also do not have a good guess as to what the equilibrium will be, so this suggests that we need to solve this game using the "best response" approach. Let's fix firm 2's strategy as q_2. What is Firm 1's best response? Firm 1 is trying to maximize profits, and the profit Firm 1 will get if it chooses q and Firm 2 chooses q_2 is given by:

$$\begin{aligned}\pi_1(q; q_2) &= P * q - C * q \\ &= (A - q - q_2)q - Cq\end{aligned}$$

This can look a little intimidating as there are a lot of letters. But pay attention to which of the letters are constants. We have fixed Firm 2's choice, q_2. Firm 1 is trying to figure out the best response to q_2, so q_2 does not change. It is a constant. Also, A and C are parameters of the problem. They are constants. Even though there are many letters, the only variable in this equation is q. Firm 1 is trying to determine which quantity maximizes profit. To determine the maximum of an equation, we take the derivative and set it equal to zero. The variable q appears several times, so we need to use the product rule.

$$\begin{aligned}\pi_1'(q; q_2) &= (-1)q + (A - q - q_2)(1) - C \\ &= A - q_2 - C - 2q\end{aligned}$$

When we set the derivative equal to zero and solve for q, we get that profit is maximized where $q = \frac{A-q_2-C}{2}$. This is our best response! If Firm 2 chooses q_2, this is the quantity for Firm 1 that will maximize its profits. Therefore,

$$BR_1(q_2) = \frac{A - q_2 - C}{2}.$$

We can do the same analysis for Firm 2. The problems are symmetric and we will find that Firm 2's best response to Firm 1 is:

$$BR_2(q_1) = \frac{A - q_1 - C}{2}.$$

A Nash equilibrium is a mutual best response. Therefore, it is a choice (q_1^*, q_2^*) such that:

$$BR_1(q_2^*) = q_1^* \text{ and } BR_2(q_1^*) = q_2^*.$$

We have already determined what the best response are. Therefore, we know that:

$$q_1^* = \frac{A - q_2^* - C}{2} \tag{2.1}$$

$$q_2^* = \frac{A - q_1^* - C}{2} \tag{2.2}$$

Plugging equation 2.2 into equation 2.1 we find that

$$q_1^* = \frac{A - \frac{A - q_1^* - C}{2} - C}{2}.$$

After some algebra, we find that:

$$q_1^* = \frac{A - C}{3}.$$

We can plug this into Firm 2's best response function, and we find that:

$$q_2^* = \frac{A - C}{3}.$$

Therefore, the Nash equilibrium of this game is for both firms to choose the quantity $\frac{A-C}{3}$.

Whew, that was a lot more difficult than the other games we solved! Take a deep breath, grab a cup of coffee, and try to pay attention to the overall approach without getting lost in the details. The approach we used is exactly the same as the approach we used when we solved the simple two by two games like Battle of the Sexes or the prisoner's dilemma. In those games, we underlined the best response to each action, and then looked for a box where both payoffs were underlined. For example, go back and look at the Battle of the Sexes example (Example 2 on page 11). When we fix Elizabeth's strategy to be B, Lou can get a payoff of 2 from choosing B or 0 from choosing T. Therefore, B is Lou's best response to B. This is exactly the same step as fixing q_2 and determining the quantity for Firm 1 that maximizes its profits. That best response is just harder to calculate, but the approach is identical. In Battle of the Sexes, after underlining the best responses, we look for a mutual best response. This is a box with both payoffs underlined. In Cournot, we again find best responses first and then solve for a mutual best response. This has us solving a system of two equations with two unknowns, but the idea is the same. Pay attention to the overall approach and then each step along the way will be a bite-sized problem.

2.5.2 Bertrand's model of duopoly

This is very similar to Cournot except for now the firms will choose price instead of quantity. Two firms must simultaneously and independently decide what price to charge for their product. Consumers will buy from the firm that offers the lower price. Specifically, if the lowest price offered is P, then the firm that offered that price will get $A - P$ customers at that price (if the two firms offer the same price P, then each gets $\frac{A-P}{2}$ customers). The firm with the higher price gets no customers. Each firm has the same constant marginal cost C. What is the Nash equilibrium?

Unfortunately, we cannot solve this game using best responses. Suppose the costs are 20. What is Firm 1's best response if Firm 2 chooses 60? 59.9 is a good choice. But 59.999 is a better choice. Is that the best choice? Of course not: 59.999999 is better. Because we have a continuous choice space, there is no best response.

Therefore, we have to use the incentive to deviate approach. In other words, we take guesses and try to learn from our guesses. Don't worry too much about making "good" guesses. For example, suppose A is 100 and C is 20. Let's start with two bad choices: Firm 1 chooses 5 and Firm 2 chooses 10. Firm one is selling to 95 customers and losing 15 on every item it sells. Note that if it chooses a price of 100 (or anything greater than 10), it would sell to 0 customers and make 0. While this isn't great, it is *better* than losing money, so Firm 1 has an incentive to deviate. Interestingly, Firm 2 does not have an incentive to deviate. It is choosing a silly price (if it sold at 10 it would lose money), but it is not selling any goods at that price, so its profits are 0. If Firm 1 is choosing 5, then there is no choice 2 can make and have positive profits. We conclude that it cannot be a Nash equilibrium if both firms offer prices below 20. In fact, it cannot be a Nash equilibrium if either firm offers a price under 20.

What if both firms offer a price of 40? Either firm would do better by offering a price slightly less than 40, say 39. Instead of selling to half the market at 40, they can sell to the whole market at almost 40. The same reasoning indicates that if Firm 1's price equals Firm 2's price and both are greater than 20, then it cannot be a Nash equilibrium.

What if Firm 1 offers 30 and Firm 2 offers 40? Now both firms have an incentive to deviate. Firm 1 would do better by offering 39. Firm 2 would do better by offering 29 (or even 30). The same reasoning indicates that it cannot be a Nash equilibrium if either firm offers a price greater than 20.

This leaves us with one final case: Both firms offer a price of 20. Both sells to half the market, but since the price is equal to its costs, neither firm makes a profit. Does either have an incentive to deviate? If a firm raises its price, it sells to no customers. If a firm lowers its price, it loses money on each item it sells. Neither firm has an incentive to deviate; therefore, both firms offering a price of 20 is a Nash equilibrium.

2.6 Problems

Problem 1. This classic game is called Stag Hunt. Two hunters are considering whether to hunt a Haire (H) or a Stag (S). A hunter can catch a Haire by herself. She does not need the cooperation of the other. However, she will only successfully catch a Stag if the other hunter cooperates with her and also chooses Stag. In particular, the payoffs are

		Elizabeth	
		Stag	Hare
Lou	Stag	10,10	1,0
	Hare	0,1	1,1

What are the Nash equilibria?

Problem 2. Bob, Umut, and Thayer each must decide whether or not to attend a faculty meeting. At least two faculty members must show up in order for the meeting to happen (if 1 or 0 people show up, the meeting doesn't count). If there is a successful faculty meeting, each person gets 10 *regardless of who showed up*. Showing up at the meeting costs a person 5 points. What are the Nash equilibria?

Problem 3. Two dogs, Jerry and Duece, are fighting over a toy. For every second they continue to fight, they both incur a cost of one unit per second. The dog that fights the longest gets the prize worth 20 units. Both Jerry and Duece must pick simultaneously how long they are going to fight. For example, if Jerry picks 15 seconds and Duece picks 10 seconds, then Jerry gets a payoff of $20 - 10 = 10$ and Duece gets a payoff of $0 - 10 = -10$. What are the Nash Equilibria of this game?

Problem 4. Two people have \$10 to divide between themselves. They use the following procedure. Each person names a number of dollars (a nonnegative integer) at most equal to 10. If the sum of the amounts that the people name is at most 10, then each person receives the amount of money she named (and the remainder is destroyed). If the sum of the amounts that the people name exceeds 10 and the amounts named are different, then the person who named the smaller amount receives that amount and the other person receives the remaining money. If the sum of the amounts that the people name exceeds 10 and the amounts named are the same, then each person receives \$5. What is the best response to x where $0 \leq x \leq 10$? What are the Nash equilibria?

Problem 5. Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or to stand. Sitting alone is more comfortable than sitting next to the other person which is more comfortable than standing. Suppose the person only cares about her own comfort. Find the pure strategy Nash equilibria. Suppose the person is altruistic, ranking outcomes according to the *other* person's comfort, but, out of politeness, prefers to stand than to sit if the other person stands. Find its pure strategy Nash equilibria.

Problem 6. Player 1 and 2 must each choose a positive number. Player 1 and player 2's payoffs when 1 chooses x and 2 chooses y are:

$$
\begin{aligned}
u_1(x,y) &= (3 + 6y - \frac{3x}{2})x \\
u_2(x,y) &= (6 - x - \frac{y}{2})y
\end{aligned}
$$

What is the Nash Equilibrium? (*Hint: go take a look at how we solved Cournot.*)

Chapter 3

Mixed Strategies

Let us return to one of the games we looked at in the last chapter: Matching Pennies.

		Elizabeth	
		H	T
Lou	H	$\underline{1}$,-1	-1,$\underline{1}$
	T	-1,$\underline{1}$	$\underline{1}$,-1

We have underlined the best responses and notice there is no mutual best response. But we are only looking at some of the strategies, and we will see that when we look at all possible strategies, then we will find there is a Nash equilibrium. If you and I played rock-paper-scissors together, neither of us would always choose the same option. We would mix it up, or else we would always lose.

Definition

A **pure strategy** is one where you always play the same action.

What's a precise way of saying "mix it up"? Well, this would mean you don't always play the same action, but notice that you always have to play something. The technical term for this is a **probability distribution**. "Don't always play the same action." Mathematically, this means you play any action with a probability between 0 and 1. "You always have to play something." Mathematically, this means the probabilities for each action sum to one.

Definition

Given possible actions $a_1, \ldots, a_n$, a **mixed strategy** is a number p_i for each action a_i such that:

- $0 \leq p_i \leq 1$
- $\sum_i p_i = 1$

Almost all of our games will have very few actions, so this is simple in practice. In the matching pennies game above, an example of a mixed strategy is playing heads with probability .4 and tails with probability .6. Or playing heads with probability .7 and tails with probability .3. This is just a technical point, but a pure strategy is also a mixed strategy. Specifically, the pure strategy of always playing heads is the same as the mixed strategy where you play heads with probability 1 and tails with probability 0.

3.1 Solving for mixed strategy equilibria

We solve for all mixed strategy equilibria in the same way. The essential point is that in order for it to be optimal two randomize between two actions, the player must be indifferent between the two actions. To see this, remember that a player's strategy must be a best response to the other player's strategy. That means her strategy must maximize her expected payoff. What is her expected payoff? Suppose I play a_1 with probability .3 and a_2 with probability .7. Then my expected payoff is:

$$.3 * \text{ Expected payoff from playing } a_1 + .7 * \text{ Expected payoff from playing } a_2$$

What if these two expected payoffs are not equal? For example, what if the expected payoff from a_1 is greater than the expected payoff from a_2? If that's the case, then I could increase my expected payoff by playing a_1 more frequently. Instead of playing a_1 30% of the time, I would get more by playing it 40% of the time. I would have an incentive to deviate! Therefore, this could not be a Nash equilibrium.

Key Fact

In a mixed strategy equilibrium, a player is indifferent between any two actions she plays.

We will use this to solve for a mixed strategy equilibrium in Matching Pennies. Our normal form of the game is

		Elizabeth	
		H	T
Lou	H	1,-1	-1,1
	T	-1,1	1,-1

and I will use

Suppose Elizabeth plays H with probability p and T with probability $1 - p$. I will denote Lou's expected payoff from playing H by $\pi_L(H)$. This payoff is:

$$\pi_L(H) = (p)(1) + (1 - p)(-1).$$

In words, her expected payoff from playing heads is the probability Elizabeth plays heads (p) times her payoff from playing heads when Elizabeth plays heads (1) plus the probability Elizabeth plays tails (1-p) times her payoff when Elizabeth plays tails and she plays heads (-1).

Lou's payoff from playing tails is:

$$\pi_L(T) = (p)(-1) + (1 - p)(1).$$

In any mixed strategy equilibrium, she must be indifferent between her two options. Therefore, the payoff from her two possible actions must be the same. Setting these two equations equal to each other and doing some algebra we find:

$$\begin{aligned}
\pi_L(H) &= \pi_L(T) \\
p - 1 + p &= -p + 1 - p \\
2p - 1 &= 1 - 2p \\
4p &= 2 \\
p &= \frac{1}{2}
\end{aligned}$$

Let us restate what we have found. When Elizabeth plays heads with probability $\frac{1}{2}$, then Lou has the same expected payoff from playing heads or tails. Therefore, what would be a best response to Elizabeth playing heads with probability $\frac{1}{2}$? Anything! Whatever Lou does, she has the same expected payoff!

We do the same exercise to find what will make Elizabeth indifferent. Suppose Lou plays H with probability q. Elizabeth's expected payoff from playing heads is:

$$\pi_E(H) = (q)(-1) + (1-p)(1),$$

and her expected payoff from tails is:

$$\pi_E(T) = (q)(1) + (1-p)(-1)$$

Therefore, Elizabeth gets the same expected payoff from heads and tails if:

$$\begin{aligned}
\pi_E(H) &= \pi_E(T) \\
-q + 1 - q &= q - 1 + q \\
2 &= 4q \\
q &= \frac{1}{2}
\end{aligned}$$

To restate, if Lou plays heads with probability $\frac{1}{2}$, then Elizabeth is indifferent between all of her options. Any strategy is a best response to this strategy, and in particular, playing heads with probability $\frac{1}{2}$ is a best response. We have already found that if Elizabeth plays heads with probability $\frac{1}{2}$, then Lou is indifferent between her options. Therefore anything is a best response, and in particular, playing heads with probability $\frac{1}{2}$ is a best response. These strategies are mutual best responses, therefore they constitute a Nash equilibrium.

In matching pennies, it is a Nash equilibrium for each player to play heads with probability $\frac{1}{2}$ (and tails with probability $\frac{1}{2}$).

3.1.1 Do all games have a mixed strategy equilibrium?

Consider the first game we looked at: the prisoner's dilemma.

Lou \ Elizabeth	Q	F
Q	3,3	0,4
F	4,0	1,1

Suppose Elizabeth plays Q with probability p and F with probability $1 - p$. Lou's payoff from Q is:

$$\pi_L(Q) = (p)(3) + (1 - p)(0).$$

Lou's payoff from Q is:

$$\pi_L(F) = (p)(4) + (1 - p)(1).$$

We set these payoffs equal to each other and solve for the p that makes Lou indifferent.

$$\begin{aligned} 3p &= 4p + 1 - p \\ 3p &= 3p + 1 \\ 0 &= 1 \end{aligned}$$

Well, that was unexpected. Obviously $0 \neq 1$, so what does this mean? It means there is no p that makes Lou indifferent, or in other words, it is impossible for Lou to be indifferent between Q and F. If we look closer at the payoffs, we see that F gives Lou higher payoffs than L *no matter what Elizabeth plays*. We call this a **strictly dominant strategy**. If there is a strictly dominant strategy, the there cannot be a mixed equilibrium.[1]

[1] This statement is not actually true. Player 1 could have a strictly dominant strategy and player 2 could be indifferent between two options; therefore, there could exist a mixed strategy equilibrium where Player 1 plays the same thing while Player 2 randomizes between her two choices. But this is a technicality and is not a very interesting game.

3.2 Problems

Problem 7. In the Stag Hunt from the last set of problems, find the mixed strategy equilibrium. The normal form of the game is:

		Elizabeth	
		Stag	Hare
Lou	Stag	10,10	1,0
	Hare	0,1	1,1

Problem 8. Find the mixed strategy equilibrium for Battle of the Sexes. The normal form of the game is:

		Elizabeth	
		B	T
Lou	B	2,1	0,0
	T	0,0	1,2

Problem 9. Find all the equilibria (mixed and pure) of the following game:

		Elizabeth: H	Elizabeth: T
Lou	H	2,1	1,2
	T	1,4	2,5

Problem 10. Find all the equilibria (mixed and pure) of the following game:

		2: L	2: R
1	T	1,1	3,3
	M	2,4	5,5
	B	3,2	1,1

Problem 11. Bob, Umut, and Thayer each must decide whether or not to attend a faculty meeting. At least one faculty member must show up in order for the meeting to happen. If there is a successful faculty meeting, each person gets 10 *regardless of who showed up*. Showing up at the meeting costs a person 5 points. What are the Nash equilibria?[2]

[2] I am asking for the *symmetric* equilibrium, meaning all people play the same strategy. This gamer has asymmetric, mixed-strategy equilibria as well. If you want to challenge yourself, find all the equilibria of this game.

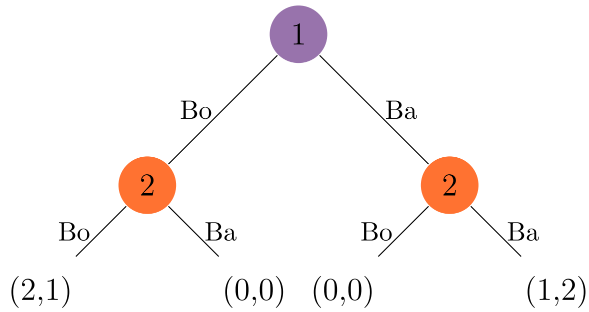
Chapter 4

Sequential Games

In every game we have looked at, a person must make her choice without knowing what choice the other person made. These are referred to as **simultaneous games**. The name - simultaneous - is a little misleading. It does not matter if the players choose at the same time or not. What is important is that each player must choose without knowing the choice of the other player. Now, we consider the alternative where one player chooses and the next player must respond. We call these **sequential games** because the players choose in sequence. Tic-tac-toe is a sequential game. You mark an X in a box, I see which box you have chosen, and I must then choose where to put my O. Connect Four is a sequential game. You place your red token in a column, and I must decide in which column to put my black token. We typically represent sequential games by game trees. Below is the simultaneous game Battle of the Sexes:

		P2	
		Boxing	Ballet
P1	Boxing	2,1	0,0
	Ballet	0,0	1,2

Now consider a sequential version of this game where Player 1 chooses Boxing or Ballet, and Player 2 learns this choice and then must choose Boxing or Ballet. We represent this in a game tree as follows:



To make sure we are on the same page, the sequence of events is as follows. Player 1 goes first. She chooses Boxing or Ballet. Player two goes second. She learns the choice Player 1 made and then has to choose either Boxing or Ballet herself. The payoffs are the same as under the simultaneous game. If Player 1 chooses Boxing and Player 2 then chooses Boxing, then Player 1 gets a payoff of 2 and Player 2 gets a payoff of one. If Player 1 chooses Boxing and Player 2 then chooses Ballet, then both players get a payoff of 0.

4.1 Backward Induction

Perhaps the most important concept you will learn in this course is backward induction. This is just a fancy way of saying "start at the end and work your way back". This is not a natural way for people to think about a problem, but it is always the right way. When faced with a difficult choice, most people ask themselves

"What should I do?"

This is actually the wrong question to ask. The right question to ask is

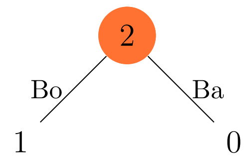
"What will happen?"

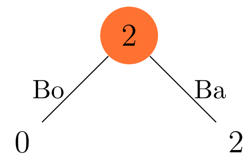
Why am I so confident that this is the right question? Because you cannot know what you should do until you know what will happen. Let's see how this works in the sequential Battle of the Sexes game. The wrong thing to do is to start with Player 1 and try to guess what Player 1 should do. The right thing to do is to figure out what will happen if Player 1 chooses "Boxing", and what will happen if Player 1 chooses "Ballet".

If Player 1 chooses Boxing, then Player 2 gets 2 if she chooses Boxing and she gets 0 if she chooses Ballet. Player 2 will choose Boxing. If Player 1 chooses Ballet, then player two gets 0 if she chooses Boxing and 1 if she chooses Ballet. Player 2 will choose Ballet. Now Player

1 knows that if she chooses Boxing, then she will get 2, and if she chooses Ballet, then she will get 1. Player 1 should choose Boxing.

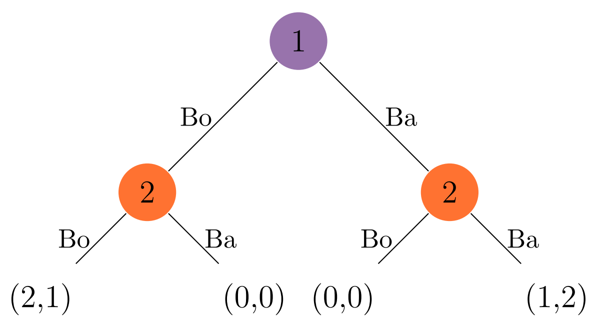
In practice, these games are solved by starting at the end and working our way back. We look at the last decision made in the game. These are made by Player 2. Remember, Player 2 only cares about her own payoff, so the decisions she faces are as follows:



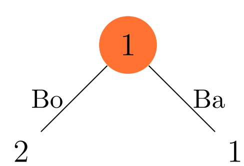


These are as trivial as games get. In the game on the left, Player 2 has to decide should she choose Boxing and get 1 or Ballet and get 0. You don't need a course on game theory to know that the equilibrium of this game is for her to choose Boxing. In the game on the right, clearly she should choose Ballet (and receive 2). This is the "backward" part of backward induction. Start at the end of the game, and work your way backward.

Initially, the decision Player 1 faced looked as follows:



But now player 1 has already determined what will happen if she chooses Boxing (the only equilibrium is for 2 to choose Boxing) and what will happen if she chooses Ballet (the only equilibrium is for 2 to choose Ballet). Using what she has already determined will happen, Player 1's decision now simplifies to:



This is the “induction” part of backward induction. Once Player 1 figures out what will happen in the game on the left, she can “collapse” the game into something much simpler. She does not need to keep track of why she will get a payoff of 2 in the game on the left. Once she has determined that choosing Boxing will result in an outcome of 2 for her, she can replace that part of the tree with the outcome: 2.

4.2 Strategies in sequential games

Our definition of an equilibrium has not changed. A **Nash equilibrium** is a strategy for each player such that no one has an incentive to deviate. But “strategies” are more complex in sequential games than in simultaneous game. A strategy is your plan. To emphasize, a strategy is **not** what you do. A strategy is your plan for what you would do in every possible situation. In practice, for a sequential game, a strategy is the choice you plan on making at every one of your nodes that could possibly be reached. In the Battle of the Sexes game, there is only one Player 1 node, the first one. Therefore, Player 1 has two possible strategies: Boxing or Ballet. So far this looks the same as for simultaneous games. We see the difference when we look at Player 2’s possible strategies. Player 2 has two nodes, and two possible choices at each node, so she has four possible strategies. One such strategy is “If you choose Boxing, I will choose Boxing and if you choose Ballet, then I will choose Ballet.” Another strategy is “If you choose Boxing, I will choose Ballet and if you choose Ballet I will choose Boxing.” This is not a very good strategy, but it is a strategy nonetheless.

4.3 Nash equilibria in sequential games

Once we have determined all possible strategies, it is fairly straightforward to find all Nash equilibria. We simply write the strategies in a normal form game and then determine the best responses. In the Battle of the Sexes game we have been considering, Player 1 has two possible strategies: $\{Bo, Ba\}$. Player 2 has four possible strategies: $\{BoBo, BoBA, BaBo, BaBa\}$[1]. The normal form of this game is as follows.

[1] The convention I will use is strategy XY means the player chooses X in the left game and Y in the right game. So, for example, the strategy BaBo indicates the strategy “If you choose Boxing, I will choose Ballet and if you choose Ballet I will choose Boxing.”

	BoBo	BoBa	BaBo	BaBa
Bo	2,1	2,1	0,0	0,0
Ba	0,0	0,0	1,2	1,2

Now we underline best responses:

	BoBo	BoBa	BaBo	BaBa
Bo	$\underline{2},\underline{1}$	$\underline{2},\underline{1}$	0,0	0,0
Ba	0,0	$1,\underline{2}$	0,0	$\underline{2},\underline{2}$

Surprisingly, there are three Nash Equilibria:

$$\{(Bo, BoBo), (Bo, BoBa), (Ba, BaBa)\}$$

One of these we have already seen. The backward induction equilibrium is $(Bo, BoBa)$. Earlier, we said backward induction is the right way of solving sequential games. What is wrong with the other equilibria? To emphasize, we are now in the world of opinion. Nothing is "wrong" with them in the sense that they both satisfy the definition of Nash equilibrium. However, let me try to explain why game theorists are skeptical of the equilibria that are not solved via backward induction. In all other equilibria, some player is making a mistake. Equilibria are meant to predict what will happen (or at least what might happen) and we don't like prediction that rely on someone making a mistake over and over again.

For example, consider the equilibrium $(Bo, BoBo)$. Player two's strategy is "*if you choose boxing, I will choose boxing and if you choose ballet, I will choose boxing*". It would be a mistake for her to choose boxing if player one chooses boxing; she would get a strictly higher payoff by choosing ballet. This is still an equilibrium because player 1 is not choosing ballet. But I do not believe her when she says "*if player one chooses ballet, I will choose boxing*", and therefore, I do not trust this equilibrium.

The third equilibrium, $(Ba, BaBa)$, is more interesting. Part of player 2's strategy is "*if you choose boxing, I will choose ballet*". I don't believe this as this would result in a payoff of 0 instead of 1. This would be crazy. However, maybe it is crazy like a fox. If Player 1 does believe her, then her best response is to choose Ballet. Choosing Boxing would yield a payoff of 0 (if Player 2 follows through on her crazy threat) while choosing Ballet would give her a payoff of 1. In the "reasonable" equilibrium, Player 2 gets a payoff of 1. When Player 2 threatens to do something crazy ("*I will choose Ballet no matter what*"), then in this equilibrium she receives a payoff of 2. Maybe this threat isn't so crazy after all! We call this an **incredible threat**. An incredible threat is saying you are going to do something that would not be in your best interests in order to change the other person's behavior. Perhaps you have a difficult friend like this: "If we do not do the thing I want, I won't do anything at all." Perhaps you should stop being this person's friend!

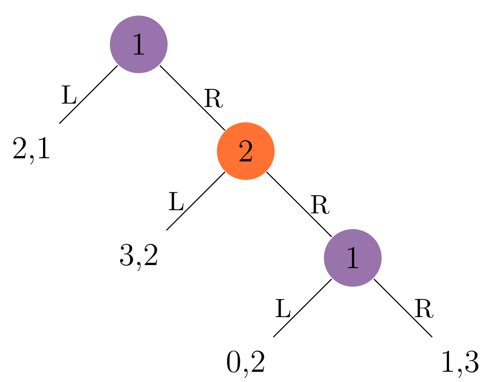
4.4 Subgame perfect equilibria

While some of these equilibria are interesting, game theorists are still skeptical of them. This motivates a new definition, subgame perfection.

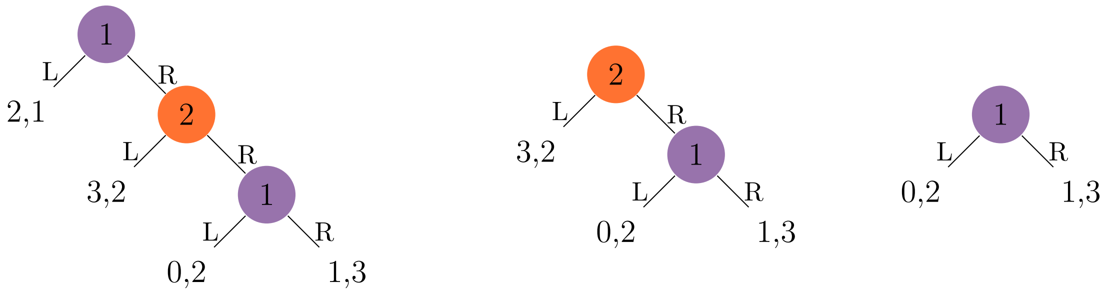
Definition

A Nash equilibrium is **subgame perfect** if it is also an equilibrium in every possible subgame.

What do we mean by "subgame"? A sequential game is the graph; the first node and all the nodes that follow. A subgame is just a subgraph; a node and what follows. For example, consider the next game.



This game has three subgames, given below.



Player 2 has two possible strategies: L or R. Player 1 has two places where (potentially) choose an action, and she has two possible actions at each choice, so Player 1 has four possible strategies: LL, LR, RL, and RR. Pause for a second and think about this as this is very confusing. Consider, for example, the strategy LR. This strategy says "*I am going to choose L and end the game. But if I had chosen R, and if 2 chooses R, then I will choose R*". This doesn't make a whole lot of sense. Player 1 is choosing to end the game. She will never play in the bottom node with this strategy, so why does she have to specify what she will do at a node that can never possibly happen?

First, it's okay if you train yourself to not ask these questions as these are questions that will only cause you confusion. Questions that only cause confusion are best not asked (at least not in the beginning). Under this approach, just accept that the definition of a strategy is a choice at every node. Why do we specify a choice at every node whether it is reached or not? Because this is our definition of a strategy, and our definition of a Nash equilibrium requires a strategy for each player. If that's not a satisfying answer–and fair enough; it is not a very satisfying answer–then think about the following. What actually happens is only part of what we are looking for. What we are really looking for is whether or not any player is making a mistake. Player 1 is ending the game (by choosing L) under this strategy, but we want to know if this is the best choice for Player 1. To know this, we need to be able to figure out what she would have received had she chosen R. What happens if she had chosen R? To know this, we must know what Player 1 will do in the last node. That is why we need a choice for every node: this is necessary for us to consider counterfactuals. If you have spent a couple of minutes thinking about this and it's still confusing, then train yourself not to ask this question and just accept that a strategy consists of a choice at every node. It will not hurt you to avoid this question.

The normal form of this game - with best responses underlined - is given below:

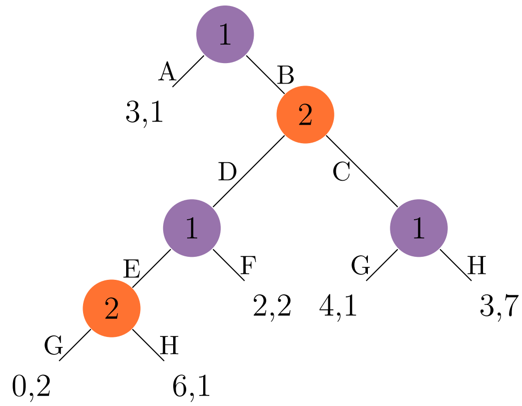
1 \ 2	L	R
LL	2,$\underline{1}$	$\underline{2}$,$\underline{1}$
LR	2,$\underline{1}$	$\underline{2}$,$\underline{1}$
RL	$\underline{3}$,$\underline{2}$	0,2
RR	$\underline{3}$,2	1,$\underline{3}$

We can see that there are three Nash equilibria: (LL,R), (LR,R), and (RL,L). The equilibrium we like we find by backward induction. In the last subgame, Player 1 should choose R (1 is better than 0). In the second to last subgame, Player 2 should choose R (3 is better than 2). Therefore, in the first game node, Player 1 should choose L (2 is better than 1). Therefore, the backward induction equilibrium is (LR,R). This equilibrium is subgame perfect.

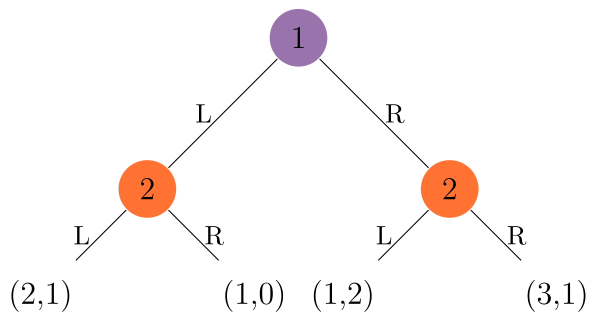
The equilibrium (LL,R) is not subgame perfect as it would be a mistake for Player 1 to choose L in the last subgame (more technically, L is not an equilibrium of the last subgame). Similarly, (RL,L) is not subgame perfect as it would be a mistake for Player 1 to choose L in the last subgame. An important point is that Player 1's first choice, R, is not a mistake when faced with Player 2's strategy of L. If this was a mistake, than Player 1 would have an incentive to deviate and this would not be a Nash equilibrium.

4.5 Problems

Problem 12. In the game below, what are all possible strategies for Player 1? Player2? Find the backward induction equilibrium (you do not need to solve for any others).



Problem 13. Find all the Nash equilibria of the following game. Which ones are subgame perfect? For those that are not, explain why not.



Problem 14. Umut and Thayer are trying to choose a movie. Thayer prefers Action to Romantic Comedy to Drama. Umut prefers a Romantic Comedy to Action to a Drama. They devise the following scheme to pick the movie. First, Thayer gets to eliminate one choice. Then Umut gets to eliminate one choice. The movie that remains is what they will watch. In equilibrium, what movie do they watch?

Problem 15. Two people have \$100 to divide between themselves. They use the following procedure. Player 1 chooses an integer between 0 and 100. Player 2 learns what Player 1 chose, and then also chooses an integer between 0 and 100. If the sum of the amounts that the people name is at most 100, then each person receives the amount of money she named (and the remainder is destroyed). If the sum of the amounts that the people name exceeds 100 and the amounts named are different, then the person who named the smaller amount receives that amount and the other person receives the remaining money. If the sum of the amounts that the people name exceeds 100 and the amounts named are the same, then each person receives \$50. What's the subgame perfect equilibrium of this game?

Problem 16. First player 1 decides (Y or N) whether or not to play the game. If she chooses N, the game ends. If she chooses Y, then Player 2 decides (Y or N) whether or not to play. If he chooses N the game ends. If he chooses Y, then they go ahead and play Battle of the

Sexes with the payoffs shown below. A player who chooses N gets 2 and the other player gets 0. What are the sub game perfect equilibria of this game?

Lou \ Elizabeth	B	T
B	1,3	0,0
T	0,0	3,1

Problem 17. A firm's output is $L(100 - L)$ when it uses L units of labor. The price of output is 1. A union that represents workers presents a wage demand (a nonnegative number w) which the firm either accepts or rejects. If the firm accepts the demand, it chooses the number of workers L to employ (which you should take to be a continuous variable, not an integer); if it rejects the demand, no production takes place. The firm's preferences are represented by its profit; the union's preferences are represented by the value of wL. What is the subgame perfect Nash equilibrium of this game?

Chapter 5

Repeated Games

Let us return to the Prisoner's Dilemma.

		Elizabeth	
		Q	F
Lou	Q	3,3	0,4
	F	4,0	1,1

This was a significant insight for economics. After Adam Smith, economists thought that individuals, acting in their own self-interest, led to an efficient outcome for the group. By efficient we mean **Pareto efficient** which is to say it is not possible to make someone better off without hurting someone else.[1] Here, we can see that when the individuals act in their own self-interest, the equilibrium is (F, F). However, (F, F) is an inefficient outcome as everyone prefers the outcome (Q, Q). In this chapter we will talk about one way individuals might be able to cooperate.

Let's pause for a second and think about the word **cooperation**. We use the word commonly, but let us be a little more careful in our usage. If I do what is best for myself and this happens to be best for you, I don't think of this as cooperation. Rather, this is an instance where our individual interests are aligned. Cooperation is when I do what is not what would be

[1] Equivalently, there is no alternative that makes everybody better off.

absolutely best for me; you do what is not what would be absolutely best for you; and the result is better than what would happen if we fight. This is exactly what we mean by *"I'll scratch your back if you scratch mine."*. I don't want to scratch your back just as you don't want to scratch mine. But we both like having our backs scratched more than we dislike scratching. The outcome (Q, Q) would be a cooperative outcome. Each of us would not be acting in our own self-interests as either player could get more by Finking. However, the outcome is better for both of us than if each follow our own self-interests.

In game theory, and economics in general, we don't trust simply saying *"The status-quo is bad. Let's cooperate!*. We think this is unstable as eventually someone will not be able to resist temptation and will cheat. We think acting in your own self-interests is stable. Why would you change? So the game theorists wonders how can we change the game so that the cooperative outcome is in each player's own self-interest. As we will see, one way is if we play the game multiple times.

What if we play the Prisoner's Dilemma 10 times instead of only once? Sadly, this does not change much. We know to solve the game by backward induction. The last time we play, there is only one equilibrium: (F, F). Now consider the second to last time. As nothing we can do in the second to last game can change the outcome of the final game, there is only one equilibrium in the second to last game: (F, F). Indeed, repeating this argument, the only subgame perfect equilibrium is for us to play F in every single period. We are not cooperating!

This is not a very satisfying answer. In fact, it is so unsatisfying that it may make you question how useful subgame perfection is. An equilibrium is only useful if it describes what people actually do. Notice that the same logic would imply that even if we played the game a thousand times, the only "reasonable" equilibrium would be for us to only ever Fink. However, this is not the way people behave in real life.

I think the problem is not with subgame perfection but rather the very question itself. We never actually play a game exactly 1,000 times. We can only use backward induction if we know exactly when will be the last time we play the game. In real life, we are never sure we will play the game exactly 1,000 times, but instead we now we will play it a lot of times. Or more precisely, we often know that we are very likely to play the game again.

5.1 Repeated Games

This motivates a new type of game which we call a **repeated game**. A repeated game is a game G that we are going to play again with a probability p. Typically, we care more about a payoff now than we do in the future. We combine these two concepts and represent it by a **discount factor** denoted by δ. In particular, if the person gets payoff π_t in each period t, then we say her payoff from playing the game is

$$\pi = \pi_0 + \delta\pi_1 + \delta^2\pi_2 + \delta^3\pi_3 + \dots$$
$$= \sum_{i=0}^{\infty} \delta^i \pi_i$$

5.2 Trigger Strategies

As their are an infinite number of periods, the players could have very complicated strategies. For our purposes, we only wish to focus on one called a **trigger strategy**. This is an intuitive strategy which says I will cooperate as long as we are cooperating, and you (or I) stop cooperating, then I will no longer cooperate. Let us write this out for the infinitely repeated Prisoner's Dilemma.

Trigger Strategy for the Prisoner's Dilemma
Period 1: Play Q
Period t: If in every earlier period we have each played Q, then play Q. If in any previous period either of us have ever played F, then F.

The first time someone does not cooperate (plays F), it "triggers" both players to stop cooperating forever after. Note that if $\delta = 0$, meaning we will not ever play the game again, then we will not cooperate. The fundamental question we ask for repeated games is how large does δ need to be in order for each player playing this trigger strategy to be a Nash equilibrium. In words, how likely must it be that we are going to play again for us to be able to sustain cooperation.

To answer this, we need to know if someone is going to cheat, when will they cheat? This is the power of infinity; it makes our analysis simple. Consider your decision in the 100^{th} period and suppose everyone has already cooperated up to this point. Your opponenet is playing a trigger strategy and you are deciding whether or not you should cheat. Everything that has already happened is in the past. You know your opponent will play Q next (since she is playing a trigger strategy and everyone has played Q up to this point). You also know that the game will be repeated an infinite number of times. All of this is to say that you face the exact same decision of whether to cheat or not in the 100^{th} period as you do in the first period. This tells us that if it is ever optimal to cheat in the future, it will be optimal to cheat immediately.

Suppose Elizabeth is playing the trigger strategy. Let's calculate when Lou's best response is to cooperate in every period. In particular, we need to know what Lou gets if she cheats and what she gets if she cooperates. From this we determine when it is better for her to cooperate than to cheat. What does Lou get if she cheats in the first period? Elizabeth is playing the trigger strategy, so she will play Q in the first period. If Lou plays F, then Lou gets 4. However, in every future period, Lou will get 1 as both players play F. Therefore, Lou's payoff from cheating is:

$$\begin{aligned}
\pi_{cheating} &= 4 + \delta * 1 + \delta^2 * 1 + \dots && (5.1)\\
&= 4 + \sum_{i=1}^{\infty} \delta^i && (5.2)\\
&= 4 + \frac{\delta}{1-\delta} && (5.3)
\end{aligned}$$

If you don't know how to calculate $\sum_{i=1}^{\infty} \delta^i$, there is a small aside at the end of the chapter on how to do it. If Lou cooperates, she gets 3 in every period as Elizabeth's strategy is to cooperate so long as Lou cooperates.

$$\begin{aligned}
\pi_{cooperating} &= 3 + \delta * 3 + \delta^2 * 3 + \dots && (5.4)\\
&= 3(1 + \delta + \delta^2 + \dots) && (5.5)\\
&= \frac{3}{1-\delta} && (5.6)
\end{aligned}$$

Cooperating is a best response so long as the payoff is greater than or equal to cheating. Mathematically, when

$$\begin{aligned}
\pi_{cooperating} &\geq \pi_{cheating}\\
\frac{3}{1-\delta} &\geq 4 + \frac{\delta}{1-\delta}\\
3 &\geq 4(1-\delta) + \delta\\
3 &\geq 4 - 4\delta + \delta\\
3 &\geq 4 - 3\delta\\
3\delta &\geq 1\\
\delta &\geq \frac{1}{3}
\end{aligned}$$

We have done a lot of math to get to where we are. We may have lost the forest for the trees. So let us pause and think about what we have found. Remember, we are trying to find if it is possible for two people, stuck in a "prisoner's dilemma", to cooperate. Our conjecture is that if the players are likely to play again and care about the future (a high δ represents both), then it will be in their self-interest to cooperate. What we have shown is that if one player is playing a trigger strategy, and $\delta \geq \frac{1}{3}$, then playing a trigger strategy is a best response to the other person's trigger strategy. In other words, the other player is going to start by cooperating, and the best response is to cooperate and to continue to cooperate so long as the other person always cooperate. Both player's cooperate!

We conclude that repeated interaction and caring about the future are essential to cooperation. More generally, what is essential is the ability to **punish**. As we know, cooperation is doing something that is good for the group even though there is a selfish option that is better for the individual. In other words, when two agents are *cooperating*, each agent faces the *temptation* to cheat. The greater the temptation, the less likely the agent will cooperate. In fact, it is only optimal to resist this temptation if the agent faces a *punishment* that is greater than or equal to the temptation itself. From a game theorist's perspective, temptation and punishment are two opposite forces that, when in balance, allow for cooperation. We can only cooperate to the extent to which we are able to punish. The greater the temptation, the less the cooperation, but the greater the punishment, the greater the ability to cooperate.

5.3 Tips for the exam

Here are a couple of points to pay attention to when taking an exam.

- *Is it greater than or greater than or equal to?* You're taking an exam and you get to the point where you are writing the equation for the payoff to cooperating needs to be more than the payoff from cheating. Wait a second, is it:

  $$\pi_{cooperating} > \pi_{cheating}$$

  or is it

  $$\pi_{cooperating} \geq \pi_{cheating}$$

  This is hard to remember, but I promise I will be looking for it as I grade the exam. This is a way for me to tell if you are rushing or if you are pausing to think about the problem. We are trying to determine if the player has an incentive to deviate. You only have an incentive to deviate if you get *more* by doing something else. If the payoff to cooperating was the same as the payoff to cheating, then the person would not have an incentive to deviate (from cooperating). Therefore, the correct expression is that cooperating will be an equilibrium so long as:

  $$\pi_{cooperating} \geq \pi_{cheating}.$$

- *Wait, the two players have different payoffs, which one is the right δ?* Consider the following slight variation to the problem:

Lou \ Elizabeth	Q	F
Q	3,3	0,5
F	4,0	1,1

The only change is the payoff when Lou is Quiet and Elizabeth Finks. Lou faces the same payoff from cooperating and cheating as in what we solved above. We have already determined that Lou will cooperate so long as $\delta \geq \frac{1}{3}$. Before you read more, as practice, try to determine on your own how high δ needs to be in order for Elizabeth to cooperate. Elizabeth's payoff from cooperating is the same as before. From Eqn (5.6) above (on page 38), Elizabeth's payoff from cooperating is:

$$\frac{3}{1-\delta}.$$

Her payoff from cheating is almost the same as before, except now she gets 5 in the first period instead of 4. Therefore, if you look at Eqn (5.3) above, you will find that her payoff from cheating is:

$$5 + \frac{\delta}{1-\delta}.$$

Elizabeth will cooperate so long as the payoff is greater than or equal to the payoff from cheating or when:

$$\begin{aligned}
\pi_{cooperating} &\geq \pi_{cheating} \\
\frac{3}{1-\delta} &\geq 5 + \frac{\delta}{1-\delta} \\
3 &\geq 5(1-\delta) + \delta \\
3 &\geq 5 - 5\delta + \delta \\
3 &\geq 5 - 4\delta \\
4\delta &\geq 2 \\
\delta &\geq \frac{1}{2}
\end{aligned}$$

Is the right answer $\delta \geq \frac{1}{3}$ or $\delta \geq \frac{1}{2}$? I find it is helpful to mentally talk through what you have found. You have found that Lou will cheat if $\delta \geq \frac{1}{3}$ and Elizabeth will cheat if $\delta \geq \frac{1}{2}$. Below $\frac{1}{3}$ they both cheat; between $\frac{1}{3}$ and $\frac{1}{2}$ Elizabeth cheats; and greater than

$\frac{1}{2}$, neither cheats. Cooperation only occurs if neither cheats, so the correct answer is the trigger strategy will be a Nash equilibrium if $\delta \geq \frac{1}{2}$.

This is not an elegant way of solving the problem. It also means you have to do lots of algebra and there are lots of chances for you to make a mistake. A better way of solving it is to think about temptation and punishment. Both players face the same benefit from cooperating (3) and face the same punishment if they cheat (1). Lou is tempted to get 4 instead of 3 by cheating. But Elizabeth is tempted to get 5 instead of 3 by cheating. Therefore, Elizabeth faces the greater temptation, and she is the one we need to worry about. If Elizabeth is willing to cooperate, then so too will Lou. So a better answer to this question is to explain that we only need to solve for Elizabeth since she is the one who faces the greater temptation.

5.4 How to calculate geometric series

In the problems of this section, we need to know how to calculate:

$$\sum_{i=0}^{\infty} \delta^i.$$

Since we don't know what it is, let's call it X. There is a cool mathematical trick which is to compare X and δX (the key is to compare it to whatever we multiply the one term by to get the next).

$$X = 1 + \delta + \delta^2 + \delta^3 + \ldots \tag{5.7}$$

$$\delta X = \delta + \delta^2 + \delta^3 + \ldots \tag{5.8}$$

These two expressions are nearly the same; X has an extra 1 relative to δX. So if we subtract (2) from (1), all that will remain is the 1.

$$\begin{aligned} X - \delta X &= 1 \\ (1 - \delta)X &= 1 \\ X &= \frac{1}{1 - \delta} \end{aligned}$$

This gives us our formula. We have indirectly determined that:

$$\frac{1}{1 - \delta} = 1 + \delta + \delta^2 + \delta^3 + \ldots \tag{5.9}$$

What about the series:

$$\delta + \delta^2 + \delta^3 + \ldots$$

We can factor out a δ and we get

$$\begin{aligned}\delta + \delta^2 + \delta^3 + \ldots &= \delta(1 + \delta + \delta^2 + \delta^3 + \ldots) \\ &= \delta(\frac{1}{1-\delta}) \\ &= \frac{\delta}{1-\delta}\end{aligned}$$

where the second line follows from Equation (5.9). Another series that is helpful to know is the following:

$$1 + \delta^2 + \delta^4 + \delta^6 + \ldots$$

We can use almost the same trick. Here, the repeating factor is δ^2, so we multiply by δ^2.

$$\begin{aligned}X &= 1 + \delta^2 + \delta^4 + \ldots \\ \delta^2 X &= \delta^2 + \delta^4 + \ldots\end{aligned}$$

Therefore,

$$\begin{aligned}X - \delta^2 X &= 1 \\ (1 - \delta^2)X &= 1 \\ X &= \frac{1}{1-\delta^2}\end{aligned}$$

This gives us our formula. We have indirectly determined that:

$$\frac{1}{1-\delta^2} = 1 + \delta^2 + \delta^4 + \ldots \tag{5.10}$$

You can memorize these formulas if you want, but it is not so easy to remember these formulas during a stressful, timed exam. You will never get it wrong if you remember how to derive it. Plus, memorizing is boring and this is a cool trick to know.

5.5 Problems

Problem 18. For the game below, write down the trigger strategy. How high must δ be for the trigger strategy to be a Nash equilibrium?

1 \ 2	A	B
A	6,6	0,12
B	12,0	1,1

Problem 19. For the game below, write down the trigger strategy. How high must δ be for the trigger strategy to be a Nash equilibrium?

1 \ 2	A	B
A	2,2	0,5
B	5,0	4,4

Problem 20. For the game below, write down the trigger strategy. How high must δ be for the trigger strategy to be a Nash equilibrium?

1 \ 2	A	B
A	6,6	0,12
B	12,0	1,4

Problem 21. For the game below, write down the trigger strategy. How high must δ be for the trigger strategy to be a Nash equilibrium?

		2	
		A	B
1	A	6,6	0,9
	B	12,0	1,1

Problem 22. This problem is harder than the others and involves a lot of, to be honest, tedious algebra. In fact, just as you don't want to do all of the algebra, I don't want to grade your algebra; therefore, this is not likely to be an exam question. But it is an interesting question, and in fact, there is a lot of economic intuition gained in solving it, so I do recommend it. Consider the classic Cournot problem where there are two firms; each firm i will choose a quantity q_i; each firm has 0 marginal costs of production; and the market price will be $P = 100 - Q$ where $Q = q_1 + q_2$. What is the Nash equilibrium of this problem? What is the maximum each firm would get if they both cooperate and then split the profits? Suppose the firms play this game an infinite number of times; how high must δ be for them to be able to cooperate and achieve the maximum possible profits?

Chapter 6

Simultaneous games with incomplete information

Up till now, each player knows everything about the game that the other knows. Not only do the players know their own payoffs, but they also know what the other person's payoffs are. In many real world games – perhaps even *most* – this is not true. Often on a job interview, if it is going well, the company will ask you "what salary are you expecting?" This is a tricky question to answer. If the number you give is too high, they won't offer you the job. If the number you give is too low, then your starting salary will be less than what it could have been. You cannot be certain how interested the company is in you nor what salary do they normally offer. This is what we mean by incomplete information - there's something important about the company that you don't know.

From our perspective, we will always view this as you are not certain what the other person's preferences are. How much is the company willing to pay for me is the same as asking does the company want to higher me a lot, a little, or not at all. This is the same if you are deciding to make an offer on a house. I do not know what other people will offer the sellers for the home. In our language, I do not know their preferences.

We will start with an example that's not realistic but designed to make things as simple as possible. Consider the classic Battle of the Sexes game:

		Elizabeth	
		B	T
Lou	B	2,1	0,0
	T	0,0	1,2

In this game, both Lou and Elizabeth know exactly how the other person feels about her. This is probably unrealistic. Perhaps a more realistic situation would be that Lou knows her own preferences, but she is unsure about whether or not Elizabeth wants to meet up with her. For simplicity, suppose that Elizabeth knows both her own preferences and Lou's. Lou believes there is a 50% chance that Elizabeth wants to meet up with her; specifically, there is a 50% chance that Elizabeth's preferences are as given above. But she also believes there is a 50% chance that Elizabeth does not want to meet up with her; specifically, there is a 50% chance that Elizabeth's preferences are as below:

		Elizabeth	
		Bo	Ba
Lou	Bo	2,0	0,1
	Ba	0,2	1,0

Note that Lou's preferences have not changed. She wants to meet up with Elizabeth and she prefers Boxing to Ballet. However, in this game, Elizabeth does not want to meet up with Lou but otherwise prefers going to the ballet over boxing. We will model this as there being two games. Lou does not know which game they are playing, but Elizabeth does. Specifically, they are playing one of two games below. Elizabeth knows which game they are playing. Lou does not, but she believes that with probability $\frac{1}{2}$ they are playing the game on the right and that with probability $\frac{1}{2}$ they are playing the game on the left.

Lou \ Elizabeth	Bo	Ba
Bo	2,1	0,0
Ba	0,0	1,2

Lou \ Elizabeth	Bo	Ba
Bo	2,0	0,2
Ba	0,1	1,0

As always, we are trying to find a Nash equilibrium. Our definition has not changed: a strategy for each player such that no player has an incentive to deviate. Just as with sequential games, the challenge is determining what a strategy is.

6.1 Strategies in simultaneous games of incomplete information

A strategy is your plan. For simultaneous games, your strategy is for every place where you could conceivably have to make a choice, what choice will you make? For simultaneous games of incomplete information, a strategy is similar.

Definition

For simultaneous games of incomplete information, a **strategy** is for every type you could be, what action will you choose.

Let's apply this to the Battle of the Sexes game above. Lou has only one type, so she there are only two possible "plans". Either she picks Boxing or she picks Ballet. Of course, she could also have a mixed strategy, but these games are complicated enough that we will never look at mixed strategy equilibria. Elizabeth has two possible types: they type that wants to meet up with Lou (the game on the left; we will label this *interested*) or the type that does not (the game on the right; we will label this *not interested*). Since she has two possible types and two possible actions for each type, Elizabeth has four possible strategies. For completeness, we will write out every possible strategy she could have:

- If I am **"interested"**, I choose **Boxing**. If I am **"not interested"**, I choose **Boxing**.
- If I am **"interested"**, I choose **Boxing**. If I am **"not interested"**, I choose **Ballet**.
- If I am **"interested"**, I choose **Ballet**. If I am **"not interested"**, I choose **Boxing**.
- If I am **"interested"**, I choose **Ballet**. If I am **"not interested"**, I choose **Ballet**.

This is mechanical. You don't need to put a lot of thought into this. You just need to mechanically list out all of the possibilities. Test yourself with the following quiz.

Quiz: A player has three possible types: $t_1, t_2,$ or t_3. For each type, she has three possible choices: A or B. How many possible strategies does she have?

Answer: She has 6 possible strategies ($3x2$). They are AAA, AAB, ABA, ABB, BAA, and BBB (where XYZ means "if I am type t_1, I choose X; if I am type t_2, I choose Y; and if I am type t_3, I choose Z.")

Now, I admit, something is strange here. The woman knows if she is interested in the man or not. If the woman is interested in the man, why does she have to say what she would have done if was not interested? This is a fair question, but for now, force yourself not to ask it. Just accept that this is what a strategy is - treat it as a definition - and later we will talk about why it is this way.

Our goal is to write this game in the normal form. Specifically, first we need to list all possible strategies for each player, and second, we need to list the payoff each player will get when they play that strategy. Each player has a type, but we need each player's expected payoff across all types. The reason is we don't want, in equilibrium, any type to be consistently making a mistake. Let us find the expected payoff for a couple of combination of strategies.

Suppose Lou is playing strategy Bo and Elizabeth is playing strategy BoBa (i.e. when she is interested, she chooses Bo and when she is not, she chooses Ba). What is Lou's expected payoff? Half the time Elizabeth is interested and she gets 2, but the other half of the time Elizabeth is not and she gets 0. Therefore, her expected payoff is 1. For Elizabeth, half the time she is interested and gets a payoff of 1 and half the time she is not interested and gets a payoff of 1. Her expected payoff is 1.

Suppose Lou is playing strategy Bo and Elizabeth is playing strategy BaBa. What is Lou's expected payoff? Elizabeth always chooses Ba, so Lou always gets 0. Therefore, her expected payoff is 0. For Elizabeth, half the time she is interested and gets a payoff of 0 and half the time she is not interested and gets a payoff of 2. Her expected payoff is 1. Bo is not a very good strategy for Lou if she thinks Elizabeth will always choose BaBa, but at this point, we are not considering what is sensible or not. We are just mechanically solving for what the expected payoffs are if the players employ these strategies.

The hard part of these games is getting the strategies right and finding the correct expected payoffs. Once you put it in the normal form, then solving for equilibria are easy.

		E			
		BoBo	BoBa	BaBo	BaBa
L	Bo	$\underline{2}$,.5	$\underline{1}$,$\underline{1.5}$	$\underline{1}$,0	0,1
	Ba	0,.5	.5,0	.5,$\underline{1.5}$	$\underline{1}$,1

6.2 Threshold strategies

In more realistic situations, there are many different types a player could be. For example, in poker you opponent could have any two card (other than the two you have!). A strategy remains the same: what the player will do for any possible combination of cards they could have. In this section we look at a type of strategy that comes up a lot, **threshold strategies**. It is not important that you remember the name, but it is important that you keep this in the back of your mind as a common type of strategy. This shouldn't be hard though as it is a natural strategy. Basically, it says if you have a high value, you will do something, and if you have a low value, you won't do something. The threshold is just the point where you switch from not doing it to doing it. This is easiest to see with examples.

Example 4. Each of two individuals receives a ticket on which there is an integer from 1 to 40 indicating the size of a prize she may receive. The individuals' tickets are assigned randomly and independently; each integer occurs with equal probability. Each individual is given the option of exchanging her prize for the other individual's prize; the individuals are given this opportunity simultaneously. If both individuals wish to exchange, then the prizes are exchanged; otherwise each individual receives her own prize. Each individual's objective is to maximize her expected monetary payoff. Solve for a Nash equilibrium of this game.

A strategy is the player's plan for what she will do with each possible number she could get. There really is only one "reasonable" type of plan: trade if my number is low and don't trade if my number is high. This is exactly what we mean by a threshold strategy. Suppose Player 2's strategy is to trade if her number is less than or equal to t for some number t. What is Player 1's best response to this strategy? We need to calculate what she expects to get if she decides to trade and the trade is successful. Player 2 will only trade if her value is less than or equal to t, and since each of these numbers are equally likely, Player 1 expects to get $\frac{t}{2}$.[1] Therefore, she is willing to trade if her value is less than or equal to $\frac{t}{2}$, but otherwise, she is

[1] This is not exactly right - her expectation is actually $\frac{t+1}{2}$ but it is not actually important that we get the expectation exactly right. What is important is that the she will respond to the threshold strategy by being willing to trade only half as much herself.

not. Notice that Player 1's best response is itself a threshold strategy. Specifically, the best response to threshold strategy t is the threshold strategy $\frac{t}{2}$. This means the game unravels. There are two equilibrium strategies that are effectively equivalent. Either the player can opt to never trade, or she can opt to trade only if her value is 1. Any combination of these two strategies is a Nash equilibrium. So, for example, both players never trade is a Nash equilibrium.

This next example is challenging. Only devote time to it after you have mastered the examples above and the practice problems.

Example 5. Two faculty members in the economics department both want to recruit a top graduate student to their department. Either faculty member can ensure the student will accept the offer by getting on the phone and shamelessly promoting the graduate program. However, there is some cost to making this call. Assume the payoffs can be represented as follows

1 \ 2	C	D
C	$1-c_1, 1-c_2$	$1-c_1, 1$
D	$1, 1-c_2$	0,0

The faculty members choose their actions simultaneously and the faculty have private information about their costs of making the call. That is, faculty member i knows c_i and believes that c_j is a random draw from a uniform distribution on $[0, 1]$. What is the Bayesian-Nash equilibrium of this game?

Obviously the professor will call if her value is 0 and won't call if her value is 1, so there must be some intermediate value where she stops being willing to call. Suppose professor 2 calls if her value is less than or equal to t. Note that this is a threshold strategy. What is the best response to threshold t? Suppose professor 1 has cost c_1. Her payoff to calling does not depend on whether or not professor 2 calls, it is simply:

$$\pi_1(C) = 1 - c_1$$

If she does not call, then she gets 1 if professor 2 calls and 0 if she does not. Therefore, we need to calculate the probability 2 calls. We know her strategy - call if $c_2 \leq t$ - and we know the distribution of her costs - c_2 is uniformly distributed between 0 and 1. Therefore, the probability she calls is the probability $c_2 \leq t$ which is equal to t. Therefore,

$$\begin{aligned} \pi_1(D) &= (\text{Probability 2 calls}) * 1 \\ &= t \end{aligned}$$

Therefore, when c_1 is low she should call and when it is high she should not. The place where she switches is the place where she is indifferent between calling and not calling. She is indifferent when:

$$\begin{aligned} \pi_1(D) &= \pi_1(C) \\ t &= 1 - c_1 \\ c_1 &= 1 - t \end{aligned}$$

Therefore, if 2 plays threshold strategy t, then 1 should call if $c_1 \leq 1 - t$. Note that the best response to a threshold strategy is a threshold strategy! We are looking for a mutual best response. Let t_1 be 1's threshold and t_2 be 2's threshold. We know that $BR_1(t_2) = 1 - t_2$. Since t_1 must be a best response, we know that $t_1 = 1 - t_2$. This game is symmetric, so the same reasoning tells us that $t_2 = 1 - t_1$. The most natural solution to this is $t_1 = t_2 = \frac{1}{2}$. But notice that any combination t_1 and $t_2 = 1 - t_1$ works. For example, 1 calls if her costs are below .2 and 2 calls if her costs are below .8.

6.3 Problems

Problem 23. Suppose Elizabeth and Lou are playing one of the two normal form games below. Elizabeth knows which game is being played, but Lou thinks that it is the game on the left with probability $\frac{1}{2}$ and the game on the right with probability $\frac{1}{2}$. Find a pure strategy Nash equilibrium of this game.

Lou \ Elizabeth	A	B
A	0,0	2,2
B	4,4	0,0

Lou \ Elizabeth	A	B
A	2,2	0,0
B	0,0	1,1

Problem 24. Suppose Elizabeth and Lou are playing one of the two normal form games below. Elizabeth knows which game is being played, but Lou thinks that it is the game on the left with probability $\frac{1}{2}$ and the game on the right with probability $\frac{1}{2}$. Find a pure strategy Nash equilibrium of this game.

		Elizabeth	
		A	B
Lou	A	2,0	0,1
	B	1,1	0,0

		Elizabeth	
		A	B
Lou	A	0,1	0,0
	B	2,1	0,1

Problem 25. Melinda and Leslie are involved in a dispute. Melinda does not know if Leslie is strong or weak; she assigns probability $\frac{1}{3}$ to Leslie being strong. Leslie knows which type she is. Each person can either fight or yield. Each receives a payoff of 0 if she yields (regardless of what her opponent does) and 1 if she fights and her opponent yields. If they fight, Melinda receives 1 and Leslie -1 if Leslie is weak, and Melinda receives -1 and Leslie 1 if Leslie is strong. Both players try to maximize expected profit, and both must choose whether to fight or yield simultaneously. What is the Nash Equilibrium of this game?

Problem 26. Consider the following variation on Cournot. There are two firms, A and B. Firm A either has marginal costs equal to 0 or 2. Each is equally likely. Firm B has marginal costs equal to 0. Both firms know Firm B's marginal costs but only Firm A knows her own costs. Both must simultaneously choose a quantity to produce and the price they will receive is $10 - Q$ where $Q = Q_A + Q_B$, i.e. the total quantity chosen by both firm A and B. What is the Nash Equilibrium of this game?

Problem 27. Two faculty members in the economics department both want to recruit a top graduate student to their department. Either faculty member can ensure the student will accept the offer by getting on the phone and shamelessly promoting the graduate program. However, the benefit to having the student is different for the professors. Assume the payoffs can be represented as follows

		2	
		C	D
1	C	b_1, b_2	$b_1, .25 + b_2$
	D	$.25 + b_1, b_2$	.25,.25

Assume the faculty members choose their actions simultaneously and the faculty have private information about their benefit. That is, faculty member i knows b_i and believes that b_j is a random draw from a uniform distribution on $[0, 1]$. What is a Nash equilibrium of this game?

Problem 28. Two people are participating in a strange auction. You are only allowed to bid 0 or $\frac{1}{3}$. If you win, you must pay your bid. If both people bid the same amount, then the winner will be chosen randomly (i.e. each person has a 50% chance of winning the item). Each person knows her own valuation for the good, but does not know the other person's value is. However, each believes the other person's value is uniformly distributed between 0 and 1. What is the Nash equilibrium of this game?