What You Don't Know Can Help You in School Assignment[†]

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Abstract

No strategy-proof mechanism Pareto dominates the student-proposing Deferred Acceptance mechanism (hereafter DA). However, it is unknown if a mechanism can Pareto dominate DA in equilibrium. We demonstrate a surprising result: a market designer can do better by learning less about students' preferences when making a school assignment. Specifically, we demonstrate that running DA but limiting students to only two applications always has an equilibrium (in weakly undominated, pure strategies) that Pareto dominates DA. We also show that no mechanism that Pareto improves DA with respect to submitted preferences actually Pareto improves DA in equilibrium. Therefore, such a mechanism may not improve DA in practice.

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1 Introduction

Due in large part to the contribution of economists, many U.S. school districts now allow students to choose which school they wish to attend. However, there is no perfect mechanism for making the student assignment since an assignment must balance numerous and sometimes conflicting objectives.

As described in Abdulkadiroğlu et al. (2009), the typical objective of a school choice system is efficiency, and the market designer is constrained by strategic incentives and the need to respect priorities. Unfortunately, efficiency as an objective is fundamentally incongruent with these constraints. It is well known that the student-proposing Deferred Acceptance mechanism (hereafter DA) makes a fair assignment[1] and that Pareto dominates any other fair assignment (Abdulkadiroğlu and Sönmez, 2003).[2] However, there does not exist a fair and Pareto efficient assignment mechanism (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Moreover, it is impossible for a strategy-proof mechanism to Pareto improve the student-optimal fair assignment.[3]

There is no strategy-proof mechanism that Pareto dominates DA, but is there a manipulable mechanism which improves DA in equilibrium? Following Ergin and Sönmez (2006), we model a mechanism as a simultaneous-move game. We seek a mechanism that always has a pure-strategy Nash equilibrium such that the associated assignment weakly Pareto dominates the DA assignment. We only consider equilibria in which no student plays a weakly dominated strategy.

At first glance, one might think the correct approach is to ask the students for their

[1]Following the convention in the literature, we define an assignment as *fair* if there is no student i and school a such that i prefers a to her assignment and such that i has higher priority at a than one of the students assigned to a. This is also called *eliminating justified envy*.

[2]We will refer to this interchangeably as the DA assignment and the student-optimal fair assignment.

[3]Under a strategy-proof mechanism, reporting true preference over schools weakly Pareto dominates any other strategies. When students act truthfully, Kesten (2010) demonstrates that no strategy-proof and efficient mechanism Pareto dominates DA.

preferences, calculate the DA assignment, and then Pareto improve this assignment.[4] It is well known that such a mechanism cannot be strategy-proof;[5] however, the equilibrium properties of any such mechanism are unknown. Our Theorem 1 demonstrates that no such mechanism Pareto dominates DA once we solve for the equilibria. These mechanisms only dominate DA in a simplistic way. Once students play a best response, they no longer improve upon DA and in fact some students can be made worse off.

Next, we find a positive and surprising result. A surprising mechanism has superior equilibrium properties to DA: running DA but limiting each student's application to only two schools. We call this the **2-school DA**.[6] The intuition is as follows. Whenever we learn that a student i prefers school a to school b, this imposes a constraint on the designer. If we assign i to b, then we can only assign students to a that have higher priority at a than i. In order to improve upon the DA assignment, the designer must learn both a student's DA assignment and an alternative that she prefers. However, the designer does not want to learn all of the schools a student prefers to her DA assignment. Each school she reveals imposes a new constraint on the designer. When all schools are revealed, the only assignment that can satisfy all of the constraints is the DA assignment. Therefore, the designer does better by limiting the information a student is able to reveal. By allowing the student to rank only two schools, in equilibrium she lists her DA assignment second and a school she prefers first, and the outcome weakly Pareto dominates the DA assignment.[7]

[4] There are many ways of doing this. One such approach is the Efficiency Adjusted Deferred Acceptance mechanism introduced in Kesten (2010). Dur et al. (2015) characterize the class of mechanisms Pareto dominating DA mechanism.

[5] Otherwise it would be an efficient, strategy-proof mechanism that always selects the fair and efficient assignment when it exists. This would violate the impossibility theorem in Kesten (2010).

[6] Note that when students submit their true preferences, 2-school DA cannot be Pareto ranked relative to DA. It is straightforward to verify that a student can be made worse off (specifically by being unassigned), receive the same assignment, or receive a better assignment relative to the DA assignment. Therefore, our impossibility result does not apply.

[7] More precisely, in some equilibria students submit preferences in this manner. There are other equilibria in which students list a different fair assignment second, and in these equilibria, a student may be assigned to a worse school than her DA assignment.

For our equilibrium results, we have assumed complete information. This is for tractability and is the same approach taken by several other papers in the literature (for example, Pathak and Sönmez (2008), and Ergin and Sönmez (2006)). However, our results only depend on a student's DA assignment being predictable.[8] The preferences of the other students and indeed the assignments of the other students are only relevant to student i to the extent that they impact i's assignment. Therefore, in a sufficiently large market, our results will continue with imperfect information to the extent that students are able to predict what their DA assignment will be. Abdulkadiroğlu et al. (2006) provides anecdotal evidence that parents in Boston had grown quite adept at this. Moreover, the typical "cut-off" for a school is information that a school board could easily provide.

2 Relationship to the Literature

Haeringer and Klijn (2009) is the first paper to consider DA when students may only submit a limited number of applications. At first glance, our paper may seem at odds with theirs. They note that once students are limited in the number of schools they may rank, DA is no longer strategy-proof, and they study Nash equilibria of the associated preference revelation game. They find that any equilibrium assignment when students are limited to ranking k schools remains an equilibrium when students may list $l > k$ schools. This would imply that in equilibrium 2-school DA does no better than DA. The difference between our analysis and theirs is that they allow Nash equilibria where students play weakly dominated strategies. Specifically, for any equilibrium in which students list only k schools, it remains an equilibrium for every student to continue to list the same k schools even when they are allowed to list more schools.[9] This is a Nash equilibrium,

[8]Formally, if each student can predict her DA assignment, then she can play a strategy where she ranks her DA school second, and a school she prefers first. If every student plays such a strategy, then the outcome of the 2-school DA will weakly Pareto dominate the DA outcome.

[9]Loosely speaking, their proof proceeds as follows. Any student who can profitably deviate and receive school s may profitably deviate by ranking s first, and therefore would have a profitable deviation whether

but it is a weakly dominated strategy to only submit k schools when you find more than k schools acceptable. It is not true that all of the equilibria in weakly undominated strategies when students are constrained to list only k schools continue to be equilibria in weakly undominated strategies when students rank $l > k$ schools.[10]

Calsamiglia et al. (2010) is also closely related to the current paper. They run an experiment to test the impact of limiting the number of schools a student is able to rank.[11] Specifically, they find that due to miscoordination there is a decrease in efficiency and an increase in justified envy when students are constrained to only listing three schools.

Kesten (2010) has three results which motivate the current paper. Similar to Abdulkadiroğlu et al. (2009), he demonstrates that no strategy-proof and efficient mechanism can Pareto dominate DA.[12] Second, he identifies the precise source of DA's inefficiencies. Finally, he introduces a mechanism, the Efficiency Adjusted Deferred Acceptance mechanism (hereafter EADAM), that Pareto improves DA when students submit their true preferences. To the best of our knowledge, the equilibrium properties of EADAM are unknown. However, our Theorem 1 demonstrates that EADAM does not dominate DA in equilibrium.[13]

they can rank one, two, or any number of schools.

[10] Haeringer and Klijn (2009) also provide an example that demonstrates that the equilibria assignments cannot all be Pareto ranked relative to the DA assignment. Here again, our results do not conflict. Our paper finds that there always exists an equilibrium that Pareto dominates the DA assignment. Their paper notes that not all equilibria Pareto dominate DA. Note that our Theorem 3 demonstrates that all equilibria (that survive iterated elimination of weakly dominated strategies) can be Pareto ranked relative to the school-proposing deferred acceptance mechanism.

[11] Their experimental design limits students to ranking three schools, so it does not directly test our mechanism which limits students to two schools. However, we believe their findings remain informative for our mechanism.

[12] Kesten and Kurino (2013) identify maximal preference domains for which a strategy-proof mechanism can Pareto dominate DA.

[13] Our paper addresses the problem of making a fair assignment more efficient. There are several papers that have focused on making an efficient assignment fairer. Kesten (2004) introduces the Equitable Top Trading Cycles mechanism in order to reduce the number of priority violations induced by TTC. Morrill (2015) addresses the same problem by introducing the mechanism Clinch and Trade.

3 Model

In a school choice problem (Abdulkadiroğlu and Sönmez, 2003) there is a finite set of students, I, and a finite set of schools, A. Each school $a \in A$ has a finite number of available seats. Let q_a be the capacity of school a and $q = (q_a)_{a \in A}$ be the capacity vector. Let $\emptyset$ denote the option of being unassigned. Each student has strict preferences (complete, transitive, and antisymmetric relations) over all schools and being unassigned. We denote the preferences of student $i \in I$ by P_i and the preference profile of all students by $P = (P_i)_{i \in I}$. For each $i \in I$, let R_i denote the at-least-as-good-as relation associated with P_i. Let $\mathcal{P}$ be the set of all possible (strict) rankings over $A \cup \{\emptyset\}$. Each school has a strict priority order (complete, transitive, and antisymmetric relations) over all students. We denote the priority order of school $a \in A$ by $\succ_a$ and the priority profile of all schools by $\succ = (\succ_a)_{a \in A}$. Throughout the paper we fix the set of students, I, the set of schools, A, the capacity vector, q, and the priority profile, $\succ$.[14] We represent a problem by the preference profile P.

A **matching** $\mu : I \to A \cup \{\emptyset\}$ is a function such that the number of students assigned to a school does not exceed its capacity. Let $\mathcal{M}$ be the set of all matchings. Given matching μ, we denote the assignment of student i and the set of students assigned to school a by μ_i and μ_a, respectively. Next, we define the properties of a matching for a given problem P.

A matching $\mu \in \mathcal{M}$ **Pareto dominates** another matching $\nu \in \mathcal{M}$ if $\mu_i R_i \nu_i$ for each student $i \in I$ and $\mu_j P_j \nu_j$ for at least one student $j \in I$. A matching μ is **Pareto efficient** if there does not exist another matching $\nu \in \mathcal{M}$ which Pareto dominates μ. Given $\mu, \nu \in \mathcal{M}$, we write $\mu R \nu$ if for every student $i \in I$, $\mu_i R_i \nu_i$.

A matching μ is **non-wasteful** if there does not exist a student-school pair (i, a) such that $|\mu_a| < q_a$ and $a P_i \mu_i$. A matching μ is **individually rational** if $\mu_i R_i \emptyset$ for all $i \in I$. A matching μ is **fair** if there does not exist a student-school pair (i, a) where $a P_i \mu_i$ and

[14]For some results, we restrict the capacity of each school to one. We will make it clear when the capacities are restricted.

$i \succ_a j$ for some $j \in \mu_a$. A matching is **stable** if it is non-wasteful, individually rational, and fair. A stable matching μ is the **student-optimal stable matching (SOSM)** if it Pareto dominates any other stable matching.

A **mechanism** $\Phi : \mathcal{P}^{|I|} \rightarrow \mathcal{M}$ is a function that selects a matching for each problem $P \in \mathcal{P}^{|I|}$.[15] The matching selected by mechanism Φ for problem P is denoted by $\Phi(P)$ and the assignment of each student $i \in I$ and the set of students assigned to each school $a \in A$ are denoted by $\Phi_i(P)$ and $\Phi_a(P)$, respectively.

A mechanism Φ is **Pareto efficient (stable)** if for any problem P its outcome $\Phi(P)$ is Pareto efficient (stable) under problem P.

A mechanism Φ is **strategy-proof** if any student $i \in I$ cannot benefit from misreporting her preferences. Formally, Φ is strategy-proof if there does not exist a problem P, a student i, and a preference order P_i', such that $\Phi_i(P_i', P_{-i}) P_i \Phi_i(P)$ where $P_{-i} = (P_j)_{j \in I \setminus \{i\}}$. A mechanism Ψ is **manipulable** if it is not strategy-proof.

The student-proposing version of DA is defined as follows.[16] In the first round, each student proposes to her most preferred option in $A \cup \{\emptyset\}$. Each school tentatively accepts applicants up to its capacity and rejects the lowest priority applicants above its capacity. In every subsequent round, each student proposes to her most preferred option in $A \cup \{\emptyset\}$ that has not already rejected her. Each school tentatively accepts the highest priority applicants up to its capacity and rejects all others. The mechanism terminates when there are no new rejections and tentative assignments are made final. DA was first introduced in Gale and Shapley (1962). Roth and Sotomayor (1990) is an excellent resource on the properties of DA.

Each problem $P \in \mathcal{P}^{|I|}$ and mechanism Φ induce a game. Unless it is otherwise specified, the preference profile P will always refer to the students' true preferences. For expositional

[15] We only consider direct mechanisms in this paper.

[16] In Appendix A, we provide the definitions of the other mechanisms including TTC, the Boston Mechanism (BM), EADAM, and Deferred Acceptance plus Top Trading Cycles (DA+TTC).

convenience, we will refer to the game induced by P and Φ as the Φ game. For each student the strategy space is composed of the strict preference orders over $A \cup \{\emptyset\}$, i.e., $\mathcal{P}$. Then, $\mathcal{P}^{|I|}$ is the set of all possible strategy profiles. Let $\tilde{P}_i$ be the strategy of student i under strategy profile $\tilde{P} \in \mathcal{P}^{|I|}$. We consider the complete information environment such that preferences and priorities of all students are commonly known.[17] For a given strategy profile $\tilde{P}$ the outcome of the Φ game is the matching $\Phi(\tilde{P})$. Under Φ game, we say strategy $\tilde{P}_i$ **weakly dominates** strategy $\hat{P}_i$ if $\Phi_i(\tilde{P}_i, \bar{P}_{-i}) R_i \Phi_i(\hat{P}_i, \bar{P}_{-i})$ for all $\bar{P}_{-i} \in \mathcal{P}^{|I|-1}$ and $\Phi_i(\tilde{P}_i, P'_{-i}) P_i \Phi_i(\hat{P}_i, P'_{-i})$ for some $P'_{-i} \in \mathcal{P}^{|I|-1}$. Strategy $\tilde{P}_i$ is a **weakly undominated strategy** for $i \in I$ if there does not exist a strategy $\hat{P}_i$ such that $\hat{P}_i$ weakly dominates $\tilde{P}_i$. Strategy $\tilde{P}_i$ is a **weakly dominant strategy** for $i \in I$ if it weakly dominates any other strategy $\hat{P}_i$. A strategy profile $\tilde{P} \in \mathcal{P}^{|I|}$ is a **weakly undominated strategy profile** if for every $i \in I$, $\tilde{P}_i$ is a weakly undominated strategy.

Under Φ game a strategy profile $\tilde{P} \in \mathcal{P}^{|I|}$ is a **Nash equilibrium (NE)** if there does not exist a student $i \in I$ and strategy $\hat{P}_i$ such that $\Phi_i(\hat{P}_i, \tilde{P}_{-i}) P_i \Phi_i(\tilde{P})$. A NE $\tilde{P} \in \mathcal{P}^{|I|}$ is a **weakly undominated NE** if $\tilde{P}$ is a weakly undominated strategy profile. We say a matching μ is an **equilibrium assignment** for mechanism Φ if there exists a NE $\tilde{P} \in \mathcal{P}^{|I|}$ under Φ game such that $\Phi(\tilde{P}) = \mu$. Similarly, a matching μ is an **weakly undominated equilibrium assignment** for mechanism Φ if there exists a weakly undominated NE $\tilde{P} \in \mathcal{P}^{|I|}$ under Φ game such that $\Phi(\tilde{P}) = \mu$. In the rest of the paper whenever we say equilibrium and weakly undominated equilibrium we mean NE and weakly undominated NE, respectively.

Iterated elimination of weakly dominated strategies is the standard process of removing weakly dominated strategies, one by one, until no weakly dominated strategies remain. See, for example, Mas-Colell et al. (1995).

[17] Ergin and Sönmez (2006), Pathak and Sönmez (2008) and Haeringer and Klijn (2009) also assume complete information.

4 Results

We wish to compare the weakly undominated equilibrium assignments for a mechanism Φ to what the assignment would have been under DA under the true preferences. Note that since truthful revelation of preferences is a weakly dominant strategy under DA, for any problem $P \in \mathcal{P}^{|I|}$, $DA(P)$ is the unique weakly undominated equilibrium assignment of the DA game.

Definition 1. *A mechanism Φ **Pareto dominates DA in equilibrium** if for every problem $P \in \mathcal{P}^{|I|}$, under Φ game there exists a weakly undominated equilibrium $\tilde{P}$ such that $\Phi(\tilde{P})\ R\ DA(P)$, and for some problem $P' \in \mathcal{P}^{|I|}$, there exists a weakly undominated equilibrium $\hat{P}$ such that $\Phi(\hat{P})$ Pareto dominates $DA(P')$ according to P'.*[18]

If Φ Pareto dominates DA in equilibrium, this does not necessarily make Φ a superior mechanism. For some problems, mechanism Φ may have equilibrium assignments that are Pareto dominated by the DA assignment. However, it can be interpreted that in the best case scenario, Φ is a superior mechanism to DA.

There are mechanisms that Pareto improve DA relative to the submitted preferences, for instance, both Kesten's EADAM and DA + TTC. We define a mechanism Φ as **improving DA directly** if for any problem P, $\Phi(P) R\ DA(P)$ and for some problem P', $\Phi(P')$ Pareto dominates $DA(P')$. While it is known that no Pareto efficient mechanism that improves DA directly can be strategy-proof, little is known about the equilibrium properties of any such manipulable mechanism. Our first result demonstrates that no Pareto efficient mechanism that improves DA directly can Pareto dominate DA in equilibrium. Directly improving DA creates perverse incentives.

[18] More generally, we can compare any two mechanisms this way. For a mechanism Φ and problem P, let $\Lambda_\Phi(P)$ denote the set of weakly undominated equilibrium assignments. We say a mechanism Φ **Pareto dominates mechanism Ψ in equilibrium** if for every problem P and for every $\mu \in \Lambda_\Psi(P)$, either $\mu \in \Lambda_\Phi(P)$ or there exists $\nu \in \Lambda_\Phi(P)$ such that ν Pareto dominates μ according to P. Moreover, we require that for some problem $\bar{P}$, if $\mu \in \Lambda_\Psi(\bar{P})$, then there exists $\nu \in \Lambda_\Phi(\bar{P})$ such that ν Pareto dominates μ according to $\bar{P}$. Note that, $\Lambda_{DA}(P) = DA(P)$ for any problem P.

Theorem 1. *Let Φ be a Pareto efficient mechanism that improves DA directly. Mechanism Φ does not Pareto dominate DA in equilibrium.*[19]

Proof. Let Φ be any mechanism that Pareto improves DA directly. Consider a problem with three students, $I = \{i, j, k\}$, three schools each with unit capacity, $A = \{a, b, c\}$, and the following preferences, P, and priorities, $\succ$:

P_i	P_j	P_k	$\succ_a$	$\succ_b$	$\succ_c$
a	a	b	k	j	i
c	b	c	i	k	k
b	c	a	j	i	j
$\emptyset$	$\emptyset$	$\emptyset$			

We will prove below that the only weakly undominated strategies for j are $P_j : a, b, c, \emptyset$ and $\hat{P}_j : a, b, \emptyset, c$. For now, we assume j submits her true preferences P_j (an analogous argument shows the same conclusion holds if she submits $\hat{P}_j$). Since j has the highest priority at b, for any $(P'_i, P'_k) \in \mathcal{P}^2$,

$$DA_j(P'_i, P_j, P'_k) \in \{a, b\}. \tag{1}$$

Since Φ improves DA directly, $\Phi_j(P'_i, P_j, P'_k) \in \{a, b\}$. In particular, if $DA_j(P'_i, P_j, P'_k) = a$, then $\Phi_j(P'_i, P_j, P'_k) = a$. For student k, consider the preference profile $P^*_k : b, a, c, \emptyset$. Student k has the highest priority at a, therefore $DA_k(P'_i, P_j, P^*_k) \in \{a, b\}$ for any $P'_i \in \mathcal{P}$. Therefore, for any $P'_i \in \mathcal{P}$,[20] $DA(P'_i, P_j, P^*_k)$ is one of the following assignments:

$$\mu = \begin{pmatrix} i & j & k \\ c & a & b \end{pmatrix} \qquad \mu' = \begin{pmatrix} i & j & k \\ c & b & a \end{pmatrix}$$

[19] More generally, this is true for any mechanism that improves DA whenever the DA outcome is Pareto inefficient and selects the DA outcome whenever the DA outcome is Pareto efficient.

[20] More precisely, for any P'_i such that $cP'_i\emptyset$. However, an identical argument shows that if i declares c unacceptable, then Φ must assign i, j, and k to $\emptyset$, a, and b, respectively. This cannot be an equilibrium under the Φ game, since i can get c by ranking it above $\emptyset$.

If $DA(P_i', P_j, P_k^*) = \mu$, then the DA assignment is Pareto efficient and therefore $\Phi(P_i', P_j, P_k^*) = \mu$. If $DA(P_i', P_j, P_k^*) = \mu'$, then DA selects a Pareto inefficient matching under problem (P_i', P_j, P_k^*). Since μ is the only possible Pareto improvement of μ', $\Phi(P_i', P_j, P_k^*) = \mu$.

This establishes two things. First, (P_i, P_j, P_k^*) is a NE under Φ game since i's report is irrelevant and j and k receive their favorite school.[21] Second, if P^* is a NE under Φ game and $P_j^* = P_j$, then $\Phi(P^*) = \mu$ or else k would have an incentive to deviate. However, $DA(P)$ (i.e. under true preferences) is the following assignment:

$$\nu = \begin{pmatrix} i & j & k \\ a & b & c \end{pmatrix}.$$

Since ν is Pareto efficient (relative to P), $\Phi(P) = \nu$. Therefore, under any NE P^* where $P_j^* = P_j$, $DA_i(P)\ P_i\ \Phi_i(P^*)$. Note that it is irrelevant whether or not j declares c acceptable. Therefore, if $\hat{P}_j := a, b, \emptyset, c$ then it is also true that under any NE P^* where $P_j^* = \hat{P}_j$, $DA_i(P)\ P_i\ \Phi_i(P^*)$. We conclude the proof by demonstrating that P_j weakly dominates all strategies for j (other than $\hat{P}_j := a, b, \emptyset, c$).

Recall that, for any strategies $(P_i', P_k') \in \mathcal{P}^2$, $DA_j(P_i', P_j, P_k') \in \{a, b\}$ (Eq. 1). Consider any preference $P_j' \in \mathcal{P}$ where b is ranked higher than a. Again, since j has the highest priority at b, for any $(P_i', P_k') \in \mathcal{P}^2$, $DA_j(P_i', P_j', P_k')$ is either b or a school she ranks higher (under P_j'). In particular, $DA_j(P_i', P_j', P_k') \neq a$. Since $\Phi(P_i', P_j', P_k')$ is a Pareto improvement of $DA(P_i', P_j', P_k')$ (relative to (P_i', P_j', P_k')), $\Phi_j(P_i', P_j', P_k') \neq a$. Therefore, for any $(P_i', P_k') \in \mathcal{P}^2$, $\Phi_j(P_i', P_j, P_k') R_j \Phi_j(P_i', P_j', P_k')$. Let $(\tilde{P}_i, \tilde{P}_k)$ be the degenerate preference profile where both i and k declare no schools to be acceptable. Since for any $(P_i', P_k') \in \mathcal{P}^2$, $\Phi_j(P_i', P_j, P_k') R_j \Phi_j(P_i', P_j', P_k')$, and for $(\tilde{P}_i, \tilde{P}_k)$ $\Phi_j(\tilde{P}_i, P_j, \tilde{P}_k) = a$ but $\Phi_j(\tilde{P}_i, P_j', \tilde{P}_k) \neq a$, we conclude that P_j weakly dominates any P_j' where b is ranked ahead of a.

Next, we show that P_j weakly dominates $P_j' := a, c, b, \emptyset$. Fix any $(P_i', P_k') \in \mathcal{P}^2$. Suppose $DA_j(P_i', P_j', P_k') = a$. Then $DA_j(P_i', P_j, P_k') = a$ (DA is strategy-proof) and consequently $\Phi_j(P_i', P_j, P_k') = \Phi_j(P_i', P_j', P_k') = a$. Now suppose $DA_j(P_i', P_j', P_k') = c$. Since j has the

[21] It is straightforward (but tedious) to verify that P_i, P_j, and P_k^* are all weakly undominated strategies.

lowest priority at c, neither i nor k prefers c (under submitted preferences (P'_i, P'_j, P'_k)) to her assignment (otherwise they would have justified envy). Consequently, j cannot be part of a Pareto improvement, and therefore, $\Phi_j(P'_i, P'_j, P'_k) = c$. Since $\Phi_j(P'_i, P_j, P'_k) \in \{a, b\}$, $\Phi_j(P'_i, P_j, P'_k)\ P_j\ \Phi_j(P'_i, P'_j, P'_k)$. The case when $DA_j(P'_i, P'_j, P'_k) = b$ is similar. Student j has been rejected by both a and c; therefore, both schools must be "holding onto" a student. Since there are only two students other than j, when j applies to b it does not reject a student. In particular, by revealed preference, neither i nor k prefers b (according to submitted preferences (P'_i, P'_j, P'_k)) to their assignment under $DA(P'_i, P'_j, P'_k)$. Consequently, $\Phi_j(P'_i, P'_j, P'_k) = b$ and therefore, $\Phi_j(P'_i, P_j, P'_k) R_j \Phi_j(P'_i, P'_j, P'_k)$. Therefore, we conclude that for any $(P'_i, P'_k) \in \mathcal{P}^2$, $\Phi_j(P'_i, P_j, P'_k) R_j \Phi_j(P'_i, P'_j, P'_k)$. Now suppose $P'_i := b, c, a, \emptyset$ and $P'_k := a, b, c, \emptyset$. Then $\Phi_j(P'_i, P'_j, P'_k) = c$ but $\Phi_j(P'_i, P_j, P'_k) = b$. This demonstrates that P_j weakly dominates $P'_j := a, c, b, \emptyset$. A similar argument shows that P_j weakly dominates $P'_j := c, a, b, \emptyset$. It is straightforward to verify that j cannot benefit from declaring one or more schools unacceptable. Therefore, P_j weakly dominates any strategy (other than $\hat{P}_j := a, b, \emptyset, c$). Therefore, in any undominated equilibrium, j plays P_j or $\hat{P}_j$. □

To the best of our knowledge, ours is the first paper to consider equilibrium analysis of mechanisms that Pareto improve DA. Prior work had considered only strategy-proof mechanisms. Hence, Theorem 1 is a far more general (and does not follow) from any previously known results. The two most prominent ways of Pareto improving DA are Kesten's EADAM and DA+TTC. Immediate corollaries of Theorem 1 state that neither of these mechanisms dominate DA in equilibrium.

Corollary 1. *DA+TTC does not Pareto dominate DA in equilibrium.*

Corollary 2. *EADAM with all students consenting does not Pareto dominate DA in equilibrium.*[22]

[22]Theorem 1 actually demonstrates something stronger. So long as at least one student consents, EADAM does not Pareto dominate DA in equilibrium. The proof of Theorem 1 only relies on the one student i consenting. It is interesting to contrast this result with Kesten's Proposition 3, which says that no student

With some restrictions on the preference domain, Kesten and Kurino (2013) introduce a strategy-proof mechanism which Pareto improves DA in certain problems and selects DA's outcome in all other problems. This does not contradict our Theorem 1 for two reasons. First, Theorem 1 applies to mechanisms that Pareto improve DA's outcome whenever possible and this is not satisfied by the mechanism introduced in Kesten and Kurino (2013). Second, Kesten and Kurino (2013) require students to have at least two acceptable schools, and we make no restriction on preferences.

Next, we consider running DA but only using each student's top two choices.[23] We demonstrate that this mechanism Pareto dominates DA in equilibrium. For tractability, we assume that each school has a capacity of one. Kesten (2010) identifies the source of DA's inefficiency. A student i can temporarily hold a seat at a school a, cause a student j to be rejected from a, but then later be rejected from a in favor of a higher ranked student. Kesten calls such a student an *interrupter*. Since i does not benefit and j is potentially harmed, this can lead to an inefficient assignment. An interrupter envies another student's assignment but she herself cannot benefit from a change to the assignment. We will refer to an objection by an interrupter as a *petty objection*.[24]

However, in order to reduce the number of interrupters, we must make it costly for a student to apply to a school. Under DA, it is costless to apply to a school. One way to do this is by limiting the number of schools a student is allowed to apply to. This gives the student the ability to express the school she "deserves" (her DA assignment) and a school she prefers, but it forces her to be judicious about which schools she applies to. We define the **2-school DA** as follows. For any problem P, the DA is run on the top two choices according to P. That is, if a student i is rejected by her top-two choices, then we do not

is harmed by consenting. This is true so long as all other students submit the same preferences. But this demonstrates that a student can be harmed by consenting in equilibrium.

[23]This can be interpreted as allowing students to rank at most two schools as acceptable.

[24]See Morrill (2016) for a full study of petty objections. Petty objections are closely related to λ-equity, introduced by Alcade and Romero-Medina (2015), reasonable stability, introduced by Cantala and Pápai (2015), and partial fairness, introduced by Dur et al. (2015).

allow her to apply her third choice and instead assign her to $\emptyset$.

Under 2-school DA and under DA, a student does not benefit from being an interrupter. However, under 2-school DA it is costly for a student to apply to a school where she is an interrupter, since she is limited in the number of proposals she may make. Since there is a cost but no benefit, in equilibrium a student does not apply to a school where she is an interrupter.[25]

Theorem 2. *When each school has a capacity of one, 2-school DA Pareto dominates DA in equilibrium.*

Proof. Consider a problem P. Let $\mu = DA(P)$. For any preference profile P', we define $2DA(P')$ to be the matching selected by 2-school DA. For any $\bar{P}_i \in \mathcal{P}$, let $\bar{P}_i(k)$ be the k^{th} school, possibly $\emptyset$, under $\bar{P}_i$. Given the DA assignment μ, we define P'_i to be a μ-*strategy* for student i as follows: if $\mu_i = P_i(1)$ or $\mu_i = \emptyset$, then $P'_i = P_i$; otherwise, $P'_i(1)\ P_i\ \mu_i$ and $P'_i(2) = \mu_i$ (i ranks her DA assignment second and a school she prefers first). We will construct a NE under 2-school DA where each student plays a μ-strategy. It is straightforward to verify the following fact.

Fact 1: Consider any preference profile $P' = (P'_i)_{i \in I}$ such that for each $i \in I$ P'_i is a μ-strategy. Then $2DA(P')\ R\ \mu$. Moreover, $2DA_j(P') = \emptyset$ if and only if $\mu_j = \emptyset$.

Under 2-school DA, we first establish that submitting true preferences is a best response for students unassigned under μ when all other students play μ-strategy.

Fact 2: Suppose $\mu_i = \emptyset$ and for each $j \neq i$ that P'_j is a μ-strategy. Then P_i is a best response to $P'_{-i} = (P'_j)_{j \neq i}$ under 2-school DA.

This argument is repeated in the proof, so we emphasize the intuition here. Let $\nu = 2DA(P_i, P'_{-i})$. Note that (P_i, P'_{-i}) is a μ-strategy profile. There may be a student j such that $\nu_j\ P_i\ \nu_i$ and where $i \succ_{\nu_j} j$. But there is no preference i can submit that will result in

[25] The exception to this is if her DA assignment is her second favorite school and she is an interrupter at her favorite school. In equilibrium, she applies to her two favorite schools.

her being assigned to ν_j when others play P'_{-i}. If i applies to ν_j under 2-school DA, then j will be rejected. Since each student (other than i) is playing a μ-strategy, by Fact 1, this initiates a rejection chain where each rejected student applies to her DA assignment, causing a new student to be rejected. This can only conclude with i being rejected from ν_j by the student assigned to ν_j under μ.

We construct a weakly undominated NE iteratively. We define P^0 as follows:

$$P_i^0 = \begin{cases} P_i & \mu_i = \emptyset \\ P_i & \mu_i = P_i(1) \\ P_i(1), \mu_i, \ldots & \textit{otherwise} \end{cases}$$

where it is understood that $P_i(1), \mu_i, \ldots$ indicates that i ranks $P_i(1)$ first, μ_i second, and ranks the remaining schools according to her true preferences P_i. Let $P^0 = (P_i^0)_{i \in I}$ and $\nu^0 = 2DA(P^0)$. Since for each student i P_i^0 is a μ-strategy, by Fact 1, $\nu^0 \ R \ \mu$.

In order to identify which students have an incentive to deviate, we construct a directed graph as follows:

- Each student i and each school a is a vertex.
- If $\mu_a \neq \emptyset$, then draw a directed edge from a to μ_a.
- Draw a directed edge from student i to $P_i(1) = a$ if $i \succ_a j$ for every student j such that either (i) j ranks a first; or (ii) $\mu_j = \emptyset$ and j ranks a first or second.

Call this directed graph G^0. Note that each school points to at most one student, and each student points to at most one school. Furthermore, by construction, each school is pointed to by at most one student. Therefore, the graph partitions the students and schools into cycles, paths, or single nodes. A student i can be a single node only if $\mu_i = \emptyset$. Since μ is non-wasteful, if $\mu_a = \emptyset$, then a will be a single node. It is straightforward to verify the following properties of ν^0: (i) a student in a cycle is assigned to the school she is pointing to (a Pareto improvement of μ if the cycle has more than one student); (ii) a student in a path is assigned to the school pointing to her (her assignment under μ); and (iii) students that are single nodes remain unassigned (these are students who are unassigned by μ).[26]

We now define the set of schools that are *achievable* for a student i under graph G^0. Intuitively, a school a is achievable for i if i would form a cycle by ranking a first fixing all the other strategies under P^0. School a is also achievable if $\mu_a = \emptyset$, but since no student desires such a school, it is not relevant for any equilibrium. For expositional convenience, we do not include these schools in the definition of achievable. Mathematically, a is achievable for a student i in G^0 if there is a path from a to i and $i \succ_a j$ for every student j that either ranks a first or leave $\mu_j = \emptyset$ and j ranks a first or second under P^0_j. By our definition, unassigned students under μ do not have any achievable school.

If there is no path from a to i under P^0, then there is no strategy P'_i that i can submit that will result in 2-school DA assigning i to a when any other student j plays P^0_j. The logic is the same as for Fact 2. If i causes a student to be rejected from a, it will initiate a rejection chain that eventually causes i to also be rejected from a. Hence, P^0 constitutes an equilibrium if no student i prefers an achievable school a to ν^0_i. If a student is in a cycle under G^0, then she is assigned to her top choice under P at $2DA(P^0)$. Hence, she cannot have a profitable deviation.

If there is a student with a profitable deviation under G^0, choose one at random and label her i_1. Label i_1's favorite achievable school a_1. Construct $P^1_{i_1}$ as ranking a_1 first, μ_{i_1} second, and the remaining schools according to her true preferences. For all students $j \neq i_1$, set $P^1_j = P^0_j$, and let G^1 be the directed graph induced by preferences P^1 as described for G^0. There are potentially several ways in which G^1 might differ from G^0. First, i_1 is now part of a cycle which we label C_1. Second, consider the case where i_1 was pointing to a school a under G^0. At a minimum, i_1 is no longer part of this path. But if other students rank a first (under P^1) then there are new students and schools connected to a.

[26]Not all unassigned students are nodes. They may point to some schools.

If P^1 is an equilibrium, then we stop. If not, then we randomly choose a student i_2 with a profitable deviation (note that $i_1 \neq i_2$ as i_1 changed to her best profitable deviation and no other student changed her report) and let a_2 be her favorite achievable school under G^1. Construct $P^2_{i_2}$ as ranking a_2 first, μ_{i_2} second, and all remaining schools according to her true preferences. For all students $j \neq i_2$, set $P^2_j = P^1_j$, and let G^2 be the directed graph induced by preferences P^2. Student i_2 is not part of C_1 since these students (other than i_1) are already getting their favorite school under P. Label the cycle i_2 creates C_2.

We first observe that it is possible for i_1 to have a new profitable deviation under P^2. Since i_2 has changed her report, there are potentially more schools that i_1 is able to form a cycle with. Student i_1 is currently part of cycle C_1. Consider the case where i_1 ranks $\bar{a}$ first (under $P^1_{i_1}$). Student i_1 is currently keeping any other student from pointing to $\bar{a}$. However, if i_1 were to change her preferences, then a new student is potentially able to point to $\bar{a}$. This means that i_1 may be able to change her report and form a new cycle. However, a key point is this could only increase the length of the path pointing to i_1, and importantly, the new cycle i_1 forms is a superset of the original cycle C_1. Therefore, we let i_1 change the school she is ranking at the top if she wants. Relabel this cycle, if there is one, as C_1. Moreover, every student in C_1 other than i_1 receives their favorite assignment. Furthermore, note that even if C_1 did change, it does not intersect with C_2. Since C_2 is a cycle, it is not part of any path to i_1. In addition, i_1's deviation cannot change i_2's achievable set of schools.

If this is an equilibrium, then we stop. If not, then we select a third student with a profitable deviation, i_3. Since i_1 and i_2 do not have any further deviations, $i_3 \notin \{i_1, i_2\}$. Since the students in $C_1 \cup C_2 \setminus \{i_1, i_2\}$ all get their first choice, $i_3 \notin C_1 \cup C_2$. Label the cycle i_3 creates C_3. The key points are that this cycle cannot intersect with C_1 or C_2 (each school points to one student and a student points to at most one school). Either i_1 or i_2 may want to change her top-ranked school, but this can only expand the cycle she is in. Since there is no path between i_1 and i_2, expanding either cycle would not effect the other. Finally, after i_1 and i_2 expand their cycle (if they want), it remains that i_1, i_2, and i_3 have no profitable deviations and all other students in C_1, C_2, and C_3 get their top choice.

Due to the finiteness of A and I, this process must eventually terminate, and when it does, no student has a profitable deviation. Hence, we have constructed a NE. In our construction, all students play a μ-strategy. Therefore, by Fact 1, the equilibrium assignment we have found weakly Pareto dominates μ.

In Appendix C, we show that any $\mu-$strategy profile is a weakly undominated strategy profile. □

In Example 1, we provide a problem and a weakly undominated equilibrium assignment under 2-school DA that Pareto dominates the DA assignment.

Example 1. There are three schools, $A = \{a, b, c\}$, each with a capacity of one; three students, $I = \{1, 2, 3\}$; and preferences and priorities as follows.

P_1	P_2	P_3	$\succ_a$	$\succ_b$	$\succ_c$
a	b	b	2	1	3
b	a	a	3	2	2
c	c	c	1	3	1
$\emptyset$	$\emptyset$	$\emptyset$			

Note that DA assigns 1 to b, 2 to a, and 3 to c. Under 2-school DA, the following strategy profile is a weakly undominated NE: 1 submits P_1; 2 submits P_2; and 3 submits $P'_3 : b, c, a, \emptyset$. This is the initial strategy profile under the equilibrium construction described in the proof of Theorem 1. In the induced equilibrium assignment 1 is assigned to a, 2 is assigned to b, and 3 is assigned to c. This equilibrium assignment Pareto dominates the DA assignment.

Due to the similarities between 2-school DA and the Shanghai mechanism (Chen and Kesten, 2015),[27] one can wonder whether the sets of weakly undominated equilibrium

[27] The formal definition of the Shanghai mechanism is given in Appendix B.

assignments under these mechanisms coincide. In Appendix B, we provide an example showing that they do not.

In general, the 2-school DA does not have a unique equilibrium and not all equilibria Pareto dominate the DA assignment. In the following example we show that for some problem P 2-school DA might have an equilbrium assignment which is worse than DA assignment under problem P.

Example 2. There are three schools, $A = \{a, b, c\}$, each with a capacity of one; four students, $I = \{1, 2, 3, 4\}$; and preferences and priorities as follows.

P_1	P_2	P_3	P_4	$\succ_a$	$\succ_b$	$\succ_c$
a	a	a	b	1	2	3
b	c	b	c	2	3	2
c	b	c	$\emptyset$	3	1	1
$\emptyset$	$\emptyset$	$\emptyset$	a	4	4	4

Note that DA assigns 1 to a, 2 to c, 3 to b, and 4 to $\emptyset$. Under 2-school DA, the following strategy profile is a weakly undominated NE: 1 submits P_1; 2 submits $P_2' : a, b, c, \emptyset$; 3 submits $P_3' : a, c, b, \emptyset$; and 4 submits P_4. In the induced equilibrium assignment 1 is assigned to a, 2 is assigned to b, 3 is assigned to c, and 4 is assigned to $\emptyset$. This equilibrium assignment is Pareto dominated by the assignment of DA under true preferences.

However, our next result shows that in all equilibria that survive iterated elimination of weakly dominated strategies no student receives worse than a fair assignment. Specifically, each student receives either their school-proposing DA assignment[28] or a school she strictly prefers.

Theorem 3. *For any problem P, let μ be an arbitrary equilibrium assignment of 2-school DA that is induced by strategies surviving iterated elimination of weakly dominated strate-*

[28] A formal definition of the school-proposing DA mechanism is given in Appendix A.

gies. Under μ no student is assigned to a school worse than her assignment under school-proposing DA.

Proof. Consider the first step of the school-proposing DA mechanism, and consider any student i that gets multiple proposals.[29] Let a be her favorite acceptable school that proposes to her in the first step, and let b be any of her other proposals (in particular, $a\ P_i\ b$). Note that she is one of the q_a and q_b highest ranked students at a and b, respectively. Hence, under 2-school DA if she ever were to list either a (or b) as her top-two choices, she will not be assigned to a school worse than a (or b) according to her submitted preferences. A key point is that i does not play the following strategies: (1) ranking b first or (2) ranking a school c that did not propose to i in the first step and b second. These strategies are weakly dominated by (1) ranking a first and (2) ranking c first and a second, respectively. A second point is that in any equilibrium assignment of 2-school DA, i will not be assigned to a school worse than a.

For the inductive step, consider the k^{th} step of the school-proposing DA mechanism. Consider a student j who is holding onto a proposal from school a and has already rejected school b. Our inductive hypothesis is that j does not rank b and does not rank another school $c \notin \{a, b\}$ first and b second.[30] In the k^{th} step, consider any student i who has received more than one proposal. Let a be her favorite of all such acceptable proposals and let b be any other proposal. By the inductive hypothesis, in any equilibrium of 2-school DA that survives iteratively eliminating weakly dominated strategies, no student who has previously rejected either a or b ever lists a or b as her first choice nor does she rank it second while not ranking the school she is not tentatively holding in step $k-1$ (they have a school that they strictly prefer and that they are sure to get into). Therefore, if i lists a (b) as the first choice, she will be assigned to a (b). Similarly, if i lists another school c first

[29]We consider a version of the school-proposing DA in which the unassigned option proposes to all students.

[30]Note that it cannot be a best response for her to list a and b but rank b ahead of a when all other students play strategies that survive the iterated elimination of weakly dominated strategies.

and lists a (b) as the second choice, she will be assigned to either c or a (b). Hence, ranking b as the first choice and another school $c \notin \{a, b\}$ first and b second is weakly dominated by ranking a first or c first and a second, respectively. Moreover, i is never assigned to a school worse than she is holding onto in the k^{th} step of the school-proposing DA mechanism in any equilibrium assignment induced by strategies the survive the iterated elimination of weakly dominated strategies.

Since this is true for any step k, in any equilibrium of 2-school DA that survives the iterated elimination of weakly dominated strategies, no student is assigned to a worse school than she receives under the school-proposing DA mechanism. □

5 Conclusion

Fairness and strategy-proofness are highly desirable properties in a mechanism. However, the most fundamental normative criterion in economics is Pareto efficiency. If it is possible to make students better off without harming anyone, then we should. It is well known that the DA mechanism makes Pareto inefficient assignments. However, in this paper, with DA, we discover that it is possible to make a Pareto improving assignment *ex-post*. Since any *ex-post* modification to the assignment inevitably changes the preference submission strategy associated with DA, prior to the current paper it was unknown whether or not it was possible to implement a Pareto improvement.

This paper demonstrates that it is indeed possible. Knowing that it is possible to implement an assignment that dominates the student-optimal fair assignment begs the following question: how should we do so?

Appendix A Definition of the Mechanisms

Boston Mechanism:

For a given problem P, BM mechanism selects its outcome through the following mechanism:

Step 1: Each student applies to her most preferred school. Each school s accepts the best students according to its priority list, up to q_s, and rejects the rest.

Step $k > 1$: Each student rejected in Step $k-1$ applies to her k^{th} choice. Each school s accepts the best students among the new applicants, up to the number of remaining seats, and rejects the rest.

School-Proposing DA Mechanism:

For a given problem P, school-proposing DA mechanism selects its outcome through the following mechanism:

Step 1: Each school s proposes to top q_s students under $\succ_s$. Each student i accepts the best proposal it gets according to P_i, and rejects the rest.

Step $k > 1$: Each school s proposes to top q_s students under $\succ_s$ who have not rejected it yet. Each student i accepts the best proposal it gets according to P_i, and rejects the rest.

Top Trading Cycles Mechanism:

For a given problem P, TTC mechanism selects its outcome through the following mechanism:

Step 0: Assign a counter to each school and set it equal to the quota of each school.

Step 1: Each student points to her most preferred school among those remaining. Each remaining school points to the top-ranked student in its priority order. Due to the finiteness there is at least one cycle.[31] Assign each student in a cycle to the school she points to and remove her. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is removed.

Step $k > 1$: Each student points to her most preferred school among the remaining ones. Each remaining school points to the student with the highest priority among the remaining ones. There is at least one cycle. Assign each student in a cycle to the school she points to and remove her. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed.

Deferred Acceptance-Top Trading Cycles Mechanism

For a given problem P, DA-TTC mechanism selects its outcome through the following mechanism:

Round DA**:** Run the DA mechanism. Update the priorities by giving the highest priorities for each school to the students assigned to it.

Round TTC: Run the TTC mechanism by using the preference profile and updated priorities.

Efficiency-Adjusted Deferred Acceptance Mechanism:

In order to define the mechanism selecting the outcome of EADAM, we first present a notion that we use in the definition. If student i is tentatively accepted by school s at some step t and is rejected by s in a later step t' of DA and if there exists another student j who is rejected by s in step $t'' \in \{t, t+1, ..., t'-1\}$, then i is called an **interrupter** for s and (i, s) is called an **interrupting pair** of step t'. Under EADAM, each student decides to consent or not. For a given problem P and consent decisions, EADAM selects its outcome through the following algorithm:

[31] A cycle is an ordered list of distinct schools and distinct students $(s_1, i_1, s_2, ..., s_k, i_k)$ where s_1 points to i_1, i_1 points to s_2, ..., s_k points to i_k, i_k points to s_1.

Round 0**:** Run the DA mechanism.

Round $k > 0$: Find the last step of the DA run in Round $k - 1$ in which a consenting interrupter is rejected from the school for which she is an interrupter. Identify all the interrupting pairs of that step with consenting interrupters. For each identified interrupting pair (i, s), remove s from the preferences of i without changing the relative order of the other schools. Rerun the DA algorithm with the updated preference profile. If there are no more consenting interrupters, stop.

Shanghai Mechanism

See Chen and Kesten (2016) for a full description of the Shanghai mechanism and it's properties. The Shanghai mechanism can be defined as iteratively repeating the 2-school DA. Each student submits a ranking over all schools.

Round 1: Run 2-school DA with each student submitting her top two schools. Finalize the assignments made in this round and adjust the school capacities accordingly.

Round k**:** Run 2-school DA with each student who has not already been assigned in a previous round submitting her $2k - 1$ and $2k$ favorite schools. Finalize the assignments made in this round and adjust the school capacities accordingly.

The algorithm concludes when either all students have been assigned or else the remaining students have been rejected by every school on their application list.

Appendix B Comparision Between Shanghai Mechanism and 2-School DA

We consider the problem presented in Example 1.

Recall that DA assigns 1 to b, 2 to a, and 3 to c. The Shanghai mechanism is defined in Appendix A. Note that 1 and 2 are guaranteed their second choice, so their weakly dominant strategy under Shanghai is to submit their true preferences. Similarly, so long as 3 ranks all three schools, she will be assigned to some school under Shanghai. As c is her least preferred school, she cannot benefit (and may harm) herself by ranking c first or second. Therefore, her only undominated strategy is to submit her true preferences or $a, b, c, \emptyset$. Under either submission, student 3 is assigned to school c under Shanghai. Therefore, under the Shanghai mechanism there exists a unique weakly undominated equilibrium assignment which coincides with the (inefficient) DA assignment: 1 is assigned to b, 2 is assigned to a, and 3 is assigned to c. On the other hand, under 2-school DA, the following strategies constitute a weakly undominated equilibrium: 1 submits P_1; 2 submits P_2; and 3 submits $b, c, a, \emptyset$. In this equilibrium 1 is assigned to a, 2 is assigned to b, and 3 is assigned to c.

Appendix C Omitted Results

Proposition 1. *For any problem P, let $\mu = DA(P)$ and $P' = (P'_i)_{i \in I}$ be a $\mu-$strategy profile. Then, P' is a weakly undominated strategy profile under 2-school DA.*

Proof. First consider any student i such that $P'_i = P_i$. If i has the highest priority at $P'_i(1) \in A$ or $P'_i(1) = \emptyset$, then $2DA_i(P'_i, \tilde{P}) = P'_i(1) = P_i(1)$ for any $\tilde{P} \in \mathcal{P}^{|I|-1}$. Now suppose i does not have the highest priority at $P'_i(1) \in A$ and $P'_i(1) \neq \emptyset$. Let $j \in I$ be the student with the highest priority at $P'_i(1)$. It is easy to verify that any strategy $\tilde{P}_i \in \mathcal{P}$ in which $P_i(1)$ is not ranked first or $P_i(1)$ and $P_i(2)$ are ranked at the top cannot weakly dominate P'_i. Let $\bar{P}_i$ be a strategy profile such that $\bar{P}_i(1) = P_i(1)$ and $\bar{P}_i(2) \neq P_i(2)$. Consider a strategy profile in which j ranks $P_i(1)$ as their first choice and all other students rank $\emptyset$ as their first choice. Denote this profile with $\bar{P}_{-i}$. Then, $2DA_i(P'_i, \bar{P}_{-i}) = P_i(2) P_i 2DA_i(\bar{P}_i, \bar{P}_{-i})$. That is, P'_i is a weakly undominated strategy.

Next consider any student i such that $P'_i \neq P_i$. Recall that $P'_i(1) P_i \mu_i$ and $P'_i(2) = \mu_i$. Since μ is stable, all students assigned to a such that $a P_i \mu_i$ under μ has higher priority than i

at a and $|\mu_a| = q_a$. Let $\tilde{P}_i$ be a strategy such that either $P'_i(1)P_i\tilde{P}_i(1)$ or $\tilde{P}_i(1)P_iP'_i(1)$ and $\tilde{P}_i(2)P_iP'_i(1)$. Let $\tilde{P}_{-i}$ be a strategy profile such that all students except the ones in $\mu_{P'_i(1)}$ rank their assignment under μ as the top choice and all students in $\mu_{P'_i(1)}$ rank $\emptyset$ as the top choice. Then, $2DA_i(P'_i, \tilde{P}_{-i}) = P'_i(1)P_i2DA_i(\tilde{P}_i, \tilde{P}_{-i})$. Let $\bar{P}_i$ be a strategy such that $\bar{P}_i(1) = P'{}_i(1)$ and $\bar{P}_i(2) \neq P'{}_i(2)$. Let $\bar{P}_{-i}$ be a strategy profile such that all students rank their assignment under μ as the top choice. Then, $2DA_i(P'_i, \bar{P}_{-i}) = \mu_iP_i2DA_i(\bar{P}_i, \bar{P}_{-i})$. Let $P'_i(1)P_i\hat{P}_i(1)$. Let $\hat{P}_{-i}$ be a strategy profile such that all students rank $\emptyset$ as their top choice. Then, $2DA_i(P'_i, \hat{P}_{-i}) = P'_i(1)P_i2DA_i(\hat{P}_i, \hat{P}_{-i})$. Let $\check{P}_i(1)P_iP'_i(1)P_i\check{P}_i(2)$. Let $\check{P}_{-i}$ be a strategy profile such that students assigned to school a with $aP_iP'_i(1)$ under μ rank a as their top choice and all other students rank $\emptyset$ as their top choice. Then, $2DA_i(P'_i, \check{P}_{-i}) = P'_i(1)P_i2DA_i(\check{P}_i, \hat{P}_{-i})$. □

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